The Yamabe Problem and Related Topics

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Abstract

The Yamabe problem was first studied by Yamabe in 1960. The solution to the problem was completed by Schoen in 1984. During this period, progress was made on the problem by Trudinger and Aubin. This thesis gives a survey on the solution of the Yamabe problem and attempts to show the interaction between geometry and analysis in this example. A sketch of the positive mass theorem is given to clarify some ideas in Schoen's proof. Two related open problems are discussed in the last chapter.

Acknowledgement

I would like to express my hearty gratitude to my supervisor Dr. H.S. Luk for his generous advice and encourgement during the preparation of this thesis.

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Basic Notation

$\mathbb{R}^n$: Euclidean n-space.

S^n : n-sphere.

ω_n : Volume of the n-sphere

M : A compact Riemannian manifold of dimension ≥ 3, if not specified.

D : Connection on M.

∇ : Induced connection on a submanifold S of M.

Δ : The Laplacian $\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \sqrt{g}\, g^{ij} \frac{\partial}{\partial x^j}$.

δ_{ij} : The Kronecker delta : $\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ if $i \neq j$.

Chapter 0 Introduction

§ 0.1 Historical Background

For many years, mathematicians have been interested in the problem of prescribing curvature on a Riemannian manifold [Ka1]. In 1960, Yamabe [Ya,p99] claimed he proved that any metric on a compact Riemannian manifold of dimension ≥ 3 is conformally equivalent to one of constant scalar curvature. He claimed that a pointwise regular solution always exists for the differential equation

$$(*) \quad \frac{4(n-1)}{n-2} \Delta u - Ru = - \tilde{R} u^{\frac{n+2}{n-2}}$$

where $n \geq 3$ is the dimension of the Riemannian manifold, R is the scalar curvature and $\tilde{R}$ is a constant. As pointed out by Goto [Ya,p viii], Yamabe's motivation came from the Poincare conjecture. Namely, if any simply connected, compact Riemannian manifold always admits a metric satisfying Einstein's equation, then the Poincare conjecture for 3 and 4 dimensional cases is true. Yamabe's "solution" of (*) is purely function theoretical. The scalar curvature was regarded as a function on the Riemannian manifold only and its geometric meanings were completely ignored.

After Yamabe's death, Trudinger, in 1968, found a mistake in Yamabe's paper. He was able to correct Yamabe's proof in case the scalar curvature is non-positive. He also gave a regularity theorem for the solution of the partial differential equation.

In 1976, Aubin showed that if dim M $\geq$ 6 and M is not conformally flat, the metric of M can be conformally deformed to one which has constant scalar curvature. He used local arguments only. Also, only the local behaviour of R at a point was employed. He made use of the norm of Sobolev imbedding to investigate the constraint that such a conformal change is possible.

In 1984, Schoen gave a proof for all remaining cases. Some ideas in his proof were originated from his earlier work with Yau on the positive mass conjecture. Moreover the Green's function played an important role in his proof. We note that another motivation of Schoen's work was to study conformally flat manifolds of positive, negative and zero scalar curvature.

§ 0.2 Arrangement of the thesis

In this thesis, we trace in detail the above efforts to solve the Yamabe problem. It is our modest goal to explore, via this well-known problem, some of the relations between geometry and analysis on a Riemannian manifold.

This thesis is arranged as follows.

Chapter 1 provides the background material and preliminary information about the problem. Most of the proofs of these general results are omitted. Trudinger's regularity theorem is given in section 1.2.

Chapter 2 presents the works of Trudinger and Aubin. In section 2.1, we give a criterion for the Yamabe problem to have a solution. A weaker version of this criterion was first given by Trudinger [Tr2,p269]. The version we give here is due to Aubin [Au2,p289]. In section 2.2, we prove the Yamabe problem in case dim $M \geq 6$ and M is not conformally flat. Both Aubin and Schoen gave a proof for this case using different approaches. We shall explore Aubin's viewpoint and point out its difference from that of Schoen.

A detail proof of Schoen's solution for all remaining cases is given in Chapter 3. The proof for the conformally flat case and the 4 and 5 dimensional case are given in section 3.1 and 3.2 respectively.

We present a sketch of Schoen and Yau's proof of the positive mass conjecture in Chapter 4. Readers may find that some arguments used in the proof of Theorem B in this chapter are quite similar to that in section 3.2. There is a conceptually different proof of the positive mass theorem due to Witten [Wi]. The readers can find a very interesting survey of both proofs by Kazdan [Ka2].

The last chapter contains some results of two related problems :

(i) the problem of prescribing scalar curvature in non-compact manifolds;

(ii) the Yamabe problem in CR manifold.

It is learned from AMS Abstract (Oct. 1985) that Lee and Parker have found a unified proof for the Yamabe problem. They claim that their proof clarifies the relation between local and global arguments, and simplifies Aubin and Schoen's arguments. Unfortunately, Lee and Parker's paper is not yet available when this thesis is finished.

Chapter 1 Tools and techniques relevant to the Yamabe problem

§ 1.1 Variational approach. The constant μ

Let (M,g) be a Riemannian manifold of dimension ≥ 3 $\bar{g} = u^{\frac{4}{n-2}} g$ be a conformal metric where u is a positive function on M. If R and $\bar{R}$ are the scalar curvatures of g and $\bar{g}$ respectively, then they are related by the partial differential equation

$$(1.1) \quad \frac{4(n-1)}{n-2}\Delta u - Ru = - \bar{R}u^{\frac{n+2}{n-2}}. \qquad \text{(see § 1.4)}$$

Therefore the Yamabe problem is equivalent to solving (1.1) on a compact Riemannian manifold where $\bar{R}$ is a constant.

Let H_1 denote the completion of the space of smooth functions on M under the norm

$$\| f \|_{H_1} = (\int_M |f|^2 \, dv)^{\frac{1}{2}} + (\int_M |Df|^2 \, dv)^{\frac{1}{2}}$$

For $2 < q \leq \frac{2n}{n-2}$, $u \in H_1$, consider the functional

$$J_q(u) = \frac{\int_M |Du|^2 + \frac{n-2}{4(n-1)} Ru^2 \, dv}{(\int_M u^q \, dv)^{\frac{2}{q}}} .$$

By the Sobolev imbedding theorem [Au5,p45] H_1 can be continuously imbedded in L_q for $2 < q \leq \frac{2n}{n-2}$. Hence $J_q(u)$ is well-defined if u is not zero almost everywhere. Moreover by the Holder inequality

$$J_q(u) \geq \frac{\frac{n-2}{4(n-1)} \inf R \; \| u \|_2^2}{\| u \|_q^2} \geq \min(0, \frac{n-2}{4(n-1)} \inf R \; V^{\frac{2}{q-2}})$$

where $V = \int_M 1 \, dv$ is the volume of M. So the value

$$\inf \left\{ J_q(u) : u \in H_1 \text{ and } u \neq 0 \right\}$$

is finite. We denote it by μ_q. In fact, since $J_q(au) = J_q(u)$ for all $a \in \mathbb{R}$ and $u \in H_1$, (1.2) $\mu_q = \inf \left\{ J_q(u) : u \in H_1 \text{ and } \| u \|_q = 1 \right\}$.

Also if u is a smooth function, the set

$$\left\{ x \in M,\ u(x) = 0 \text{ and } Du(x) \neq 0 \right\}$$

has measure zero . Therefore $\| Du \|_2 = \| D|u| \|_2$. This allows us to consider nonnegative functions only. We can rewrite μ_q as :

$$\mu_q = \inf \left\{ J_q(u) : u \in H_1 \text{ and } \| u \|_q = 1 \text{ and } u \geq 0 \right\}$$

Suppose u_q is a nontrivial function in H_1 which minimizes J_q, that is, $J_q(u_q) = \mu_q$. By the Lagrange multiplier rule [Lu,p188], there is a constant λ such that u_q also minimizes the functional $I_q(u)$ in H_1, where

$$I_q(u) = \left(\int_M | Du |^2 + \frac{n-1}{4(n-2)} Ru^2 \, dv \right) + \lambda \int_M u^q \, dv$$

For $v \in H_1$

$$0 = \frac{d}{dt} I_q(u+tv)|_{t=0} = \int_M \langle Du, Dv \rangle + \frac{n-2}{4(n-1)} Ruv \, dv + \lambda \int_M u^{q-1} v \, dv$$

Take $v = u_q$. Since $J_q(u_q) = \mu_q$, we have $\lambda = -\mu_q$ and u_q is a weak solution of

(1.3) $\Delta u_q - \frac{n-2}{4(n-1)} Ru_q = - \mu_q u_q^{q-1}$.

In particular, when $q = \frac{2n}{n-2}$, it is the differential equation we require. Therefore instead of solving equation (1.1) and (1.3) directly, we try to show the existence of a non-trivial minimizing function for J_q, $2 < q \leq \frac{2n}{n-2}$. For simplicity we shall write J for J_q and μ for μ_q when $q = \frac{2n}{n-2}$.

If u is a smooth function, $q \longmapsto \| u \|_q$ is a continous function. It can then be shown that $q \longmapsto \mu_q$ is also continuous [Au3,p7].

§ 1.2 Critical case of imbedding theorem. Trudinger's regularity theorem

The Sobolev imbedding theorem states that H_1 can be imbedded in $L_{\frac{2n}{n-2}}$ with the norm of imbedding $K(n) = 2\,\omega_n^{-\frac{1}{n}}(n(n-2))^{-\frac{1}{2}}$. Moreover for any $\varepsilon > 0$, there exists a constant $A(\varepsilon) \geq 0$ such that every $u \in H_1$ satisfies .

(1.4) $\| u \|_{\frac{2n}{n-2}} \leqslant (K(n) + \varepsilon) \| Du \|_2 + A(\varepsilon) \| u \|_2$. [Au5,p45]

If $M = \mathbb{R}^n$, then we have

(1.5) $\| u \|_{\frac{2n}{n-2}} \leqslant K(n) \| Du \|_2$. [Au5,p39]

Now we can give an upper bound for the constant μ defined in section 1.1.

Proposition 1.1 $\quad \mu \leq \frac{1}{4} n(n-2)\,\omega_n^{\frac{2}{n}}$

Proof. By (1.4), we can find a sequence of smooth functions $\{u_i\}$ such that $\| u_i \|_{\frac{2n}{n-2}} = 1$, $\| u_i \|_2 \longrightarrow 0$ and $\| Du_i \|_2 \longrightarrow \frac{1}{K(n)}$ as $i \longrightarrow \infty$. Then

$$J(u_i) = \| Du_i \|_2^2 + \frac{n-2}{4(n-1)} \int_M R u_i^2 \, dv$$

$$\leqslant \| Du_i \|_2^2 + \frac{n-2}{4(n-1)} \sup |R| \, \| u_i \|_2^2 .$$

Thus $J(u_i)$ converges to $\left(\frac{1}{K(n)}\right)^2 = \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}$ as i tends to infinity. $\mu \leq \lim_{i\to\infty} J(u_i) = \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}$. Q.E.D.

From the Kondrakov theorem [Au5,p55] we also know that the imbedding $H_1 \longrightarrow L_q$ is a compact operator when $2 < q < \frac{2n}{n-2}$ But in general it does not hold for $q = \frac{2n}{n-2}$ This makes the Yamabe's case critical. With the help of Kondrakov theorem, we shall show in Chapter 2 that there exists a minimizing function u_q for J_q for $2 < q < \frac{2n}{n-2}$. Now we give a regularity theorem of the equation (1.1) due to Trudinger [Tr1.,p271].

<u>Theorem 1.2</u> Let u be a non-negative H_1 solution of (1.1). Then u is smooth. If u is non-trivial, then u is strictly positive.

<u>Proof.</u> Let u be a weak solution of (1.1). For any $v \in H_1$, v satisfies

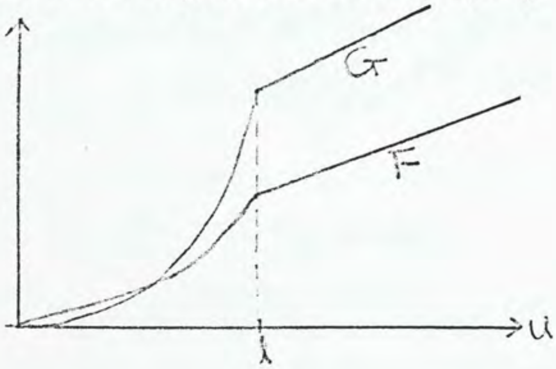
(1.6) $$\int_M \left(\langle Du, Dv\rangle + \frac{n-2}{4(n-1)} Ruv\right) dv = \mu \int_M u^{\frac{n+2}{n-2}} v\, dv$$

Define Lipshitz functions G, F

$$G(u) = \begin{cases} u^p & \text{if } u \leq \ell \\ \ell^{q-1}(q\ell^{q-1}u - (q-1)\ell^q) & \text{if } u > \ell \end{cases}$$

$$F(u) = \begin{cases} u^q & \text{if } u \leq \ell \\ q\ell^{q-1}u - (q-1)\ell^q & \text{if } u > \ell \end{cases}$$

where p, ℓ are fixed positive numbers, $2q = p+1$.



We have

(i) $F(u)^2 \geq uG(u)$,

(ii) $F'(u)^2 \leq qG'(u)$.

Let $w \geq 0$ and $w \in C^1(M)$. Substitute $w^2G(u)$ for v in (1.6). After some calculations, we have

$$\int_M w^2 G'(u)|Du|^2\, dv \leq c \int_M \left(|Dw|^2 + w^2 + u^{\frac{4}{n-2}} w^2\right) uG(u)\, dv$$

By (i) and (ii)

(1.7) $\int_M w^2|DF(u)|^2\,dv \le c_1\left(\int_M (|Dw|^2 + w^2)F(u)^2\,dv + \int_M w^2u^{\frac{4}{n-2}}F(u)\,dv\right)$

Suppose w has compact support in a coordinate patch U. U is diffeomorphic to $\{x\in\mathbb{R}^n, |x| \le r\}$ and U is chosen so that

(1.8) $\int_U |u|^{\frac{2n}{n-2}}\,dv \le \frac{1}{4c_1}$, c_1 is the constant in (1.7).

By the Holder inequality

$$\int_U w^2|DF(u)|^2\,dv \le c_1\|(|Dw| + w)F(u)\|^2_{L_2(U)} + \frac{1}{4}\|wF(u)\|^2_{L_{\frac{2n}{n-2}}(U)}$$

Then $\int_U |D(wF)|^2\,dv \le 2\int_U |Dw|^2F^2 + w^2|DF|^2\,dv$

$$\le c\|(|Dw| + w)F\|^2_{L_2(U)} + \tfrac{1}{2}\|wF\|^2_{L_{\frac{2n}{n-2}}(U)}.$$

By (1.5), $\|wF\|^2_{L_{\frac{2n}{n-2}}(U)} \le C\|D(wF)\|^2_{L_2(U)}$. It follows that

$$\|wF\|^2_{L_{\frac{2n}{n-2}}(U)} \le 2c\|(|Dw| + w)F\|^2_{L_2(U)}.$$

Suppose w is chosen so that $w = 1$ in $\{x\in\mathbb{R}^n, |x| \le \frac{1}{2}r\}$ and $|Dw| < \frac{2}{r}$. We have $\|wF\|_{L_{\frac{2n}{n-2}}(U)} \le c(1 + \frac{1}{2}r)\|F\|_{L_2(U)}$

From the definition of F, F monotonically increases to u^q as $\lambda \longrightarrow \infty$. Therefore $\|wF\|_{L_{\frac{2n}{n-2}}(U)}$ converges to $\|wu^q\|_{L_{\frac{2n}{n-2}}(U)}$ and $\|F\|_{L_2(U)}$ converges to $\|u^q\|_{L_2(U)}$. So

$$\|wu^q\|_{L_{\frac{2n}{n-2}}(U)} \le c(1 + \tfrac{1}{2}r)\left(\int_U u^{2q}\,dv\right)^{\frac{1}{2}}$$
$$\le c(1 + \tfrac{1}{2}r)\left(\int_u u^{p+1}\,dv\right)^{\frac{1}{2}}$$

Take $p = \frac{n+2}{n-2}$, $q = \frac{n}{n-2} > 1$. We obtain

(1.9) $\|wu^q\|_{L_{\frac{2n}{n-2}}(U)} \le c(1 + \frac{1}{2}r)\|u\|_{L_{\frac{2n}{n-2}}(U)}$

Since M is compact, we can find a finite covering $\{U_i\}_{i=1,2,\ldots,k}$ such that $\int_{U_i} u^{\frac{2n}{n-2}}\,dv \le c$, a fixed constant. Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i\}$. By (1.9)

$$\|u^q\|_{L_{\frac{2n}{n-2}}(M)} \le \sum_{i=1}^{k}\|\varphi_i u^q\|_{L_{\frac{2n}{n-2}}(U_i)} \le kc_2(1 + \tfrac{1}{2}r) < \infty$$

Therefore $u^q \in L_{\frac{2n}{n-2}}$ and $u^{\frac{4}{n-2}} \in L_r$, $r = q\frac{n}{2} > \frac{n}{2}$. u satisfies

$$\triangle u + u\left(\frac{-(n-2)}{4(n-1)}R - \mu u^{\frac{4}{n-2}}\right) = 0.$$

We shall use without proof the following lemma of Trudinger [Tr2, p724].

Lemma Let u be a weak, non-negative solution of the linear equation $\Delta u + fu = 0$ where $f \in L_r$, $r > \frac{n}{2}$. Then u is bounded and positive if u is non-trivial.

By this lemma, u is positive and bounded, and hence smooth by elliptic regularity theory. Q.E.D.

§ 1.3 Green's function

In $\mathbb{R}^n$, $n \geq 3$, with the standard metric, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2}$.
It is easy to see that $\Delta_y |x-y|^{2-n} = (n-2)\omega_{n-1}\delta_x(y)$. If $u \in C^2(\mathbb{R}^n)$ satisfies $\Delta u = 0$ and has a singularity of order n-2 at 0, then, up to a constant multiple, u has the form

$$u(x) = |x|^{2-n} + h(x) + A ,$$

where h is a harmonic function, $h(0) = 0$ and A is a constant. Moreover, the interior estimates of the derivatives of h [GT, p40] implies that for any x near 0, $|Dh(x)| \leq$ const.

Let M be a compact Riemannian manifold with volume V. The Green's function g of Δ is a function which satisfies

$$\Delta_y\, g(x,y) = \delta_x(y) - V^{-1} .$$

The construction of g from the distance function on M is well-known [Au5, p109]. In fact g is a smooth symmetric function on $M \times M \setminus \Delta$, where Δ is the diagonal, and for all $u \in C^2(M)$,

$$u(x) = V^{-1} \int_M u(y)\, dy + \int_M g(x,y) \Delta u(y)\, dy.$$

Consider now $L = \Delta + f$, where f is a smooth function on M. Let o be a fixed point of M and $d(x,o)$ be the distance from x to o. It is known that if Ker L = 0 , then there exists a solution of $Lu = 0$ on $M \setminus \{o\}$, which has the order $d(x,o)^{2-n}$ near o.

In particular, if $L = \Delta - \frac{n-2}{4(n-1)} R$ where $R \geq 0$ and not identically zero, then Ker L = 0. For, if $Lu = 0$, then

$$\int_M u\Delta u - \frac{n-2}{4(n-1)} Ru^2 \, dv = 0 \quad .$$

Integrating by parts, $\quad 0 \leq \int_M |Du|^2 \, dv = \int_M - \frac{n-2}{4(n-1)} Ru^2 \, dv \leq 0$,

which implies $u = 0$. By the result quoted above, there exists a function G on M such that $LG = 0$ on $M \setminus \{o\}$ and

$$\lim_{x \to o} G(x) d(x,o)^{n-2} = 1 \, .$$

By the maximum principle, G is a positive function [GT,p34]. This function G , which we shall loosely refer to as a Green's function of L, plays a crucial role in Schoen's proof.

§ 1.4 Related geometric facts

We collect here some basic facts concerning a conformal change of metric. Let φ be a positive smooth function defined on M. $\bar{g} = \varphi^{\frac{4}{n-2}} g$ is a conformal metric. $\bar{R}$ and $d\bar{v}$ are the scalar curvature and volume element relative to $\bar{g}$. Then $d\bar{v} = \varphi^{\frac{4}{n-2}} dv$ and

$4\frac{n-1}{n-2}\Delta\varphi - R\varphi = -\bar{R}\varphi^{\frac{n+2}{n-2}}$. Let

$$S_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{jk}\delta_{il} - R_{jl}\delta_{ik} + g_{jk}R_{il} - g_{il}R_{ik} + \frac{R}{(n-1)(n-2)}(g_{ik}\delta_{il} - g_{jl}\delta_{ik}),$$

$$T_{ij} = R_{ij} - R/n\, g_{ij} .$$

S_{ijkl} is called the <u>Weyl conformal tensor</u> and T_{ij} is called the <u>energy momentum tensor</u>. S_{ijkl} is a conformal invariant [Go,p115] but T_{ij} is not. Notice that $g^{ij}T_{ij} = 0$.

A space is called <u>conformally flat</u> or locally conformally flat if the Weyl conformal tensor vanishes identically. It is known that if M is conformally flat, M is locally conformal to a Euclidean space. More precisely, for each $x \in M$, there is a coordinate system centered at x such that a conformal metric $g_{ij} = \delta_{ij}$ in the coordinate patch [Go,p117].

For $u \in H_1$, let $E(u) = \int_M |Du|^2 + \frac{n-2}{4(n-1)} Ru^2\, dv$.
As before, $\bar{g} = \varphi^{\frac{4}{n-2}} g$ is a conformal metric.

$$\begin{aligned}\bar{E}(u) &= \int_M (\bar{g}^{ij}D_iuD_ju + \frac{n-2}{4(n-1)}\bar{R}u^2\, d\bar{v} \\ &= \int_M g^{ij}D_iu\, D_ju\varphi^2 + \varphi u^2(-\Delta\varphi + \frac{n-2}{4(n-1)}\bar{R}\varphi)\, dv \\ &= \int_M g^{ij}D_i(\varphi u)D_j(\varphi u) + \frac{n-2}{4(n-1)} R(\varphi u)^2\, dv \\ &= E(\varphi u) .\end{aligned}$$

Hence we have proved the following results.

<u>Proposition 1.3</u> The constant μ defined in section 1.1 is a conformal invariant.

By Proposition 1.3, we assume M has volume equal to 1 in the rest of this thesis.

Proposition 1.4   If g, of scalar curvature R, is conformal to a metric with nonnegative scalar curvature, then there exists a Green's function of the operator $Lu = \Delta u - \frac{n-2}{4(n-1)} Ru$.

Suppose Green's function G(x,y) of $Lu = \Delta u - \frac{n-2}{4(n-1)} Ru$ exists. As in section 1.3, fix a point $o \in M$ and let $G(x) = G(x,o)$. G is then a smooth function defined on $M\setminus\{o\}$ and satisfies $LG = 0$. Therefore the metric $\bar{g} = G^{\frac{4}{n-2}} g$ has constant zero scalar curvature in $M\setminus\{o\}$ by (1.1). Let $B_r(o) = \{ x \in M, \ d(x,o) < r \}$ where r is a small real number. u is a smooth function on $B_r(o)$ so that $u(x) = G(x)$ on $\partial B_r(o)$. Define

$$\varphi(x) = \begin{cases} u(x) & x \in B_r(o) \\ G(x) & \text{otherwise} \end{cases}$$

Since $LG = 0$, by Stoke's theorem

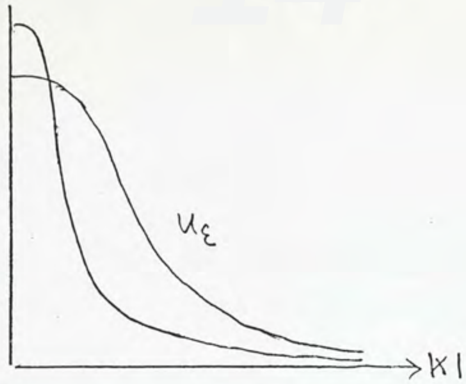
$$E(\varphi) = \int_{B_r(o)} |Du|^2 + Ru^2 \, dv - \int_{B_r(o)} \frac{\partial G}{\partial n} \, ds$$

If the asymptotic formula of G near o is known, by choosing suitable function u, we can estimate the constant μ. In Chapter 3 we shall see that Schoen employed this technique repeatedly in proving the Yamabe problem.

The simplest example.

In $\mathbb{R}^n$ with standard metric, let

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{n-2}{2}}$$



$$\frac{\partial}{\partial x^i} u_\varepsilon(x) = -(n-2)\, \varepsilon^{\frac{n-2}{2}} \left(\frac{1}{\varepsilon^2 + |x|^2}\right)^{\frac{n}{2}} x^i$$

$$\frac{\partial^2}{(\partial x^i)^2} u_\varepsilon(x) = \varepsilon^{\frac{n-2}{2}} \left(\frac{1}{\varepsilon^2 + |x|^2}\right)^{\frac{n}{2}} \left(-(n-2) + \frac{n(n-2)(x^i)^2}{\varepsilon^2 + |x|^2}\right)$$

Therefore $\Delta u_\varepsilon(x) = -\, n(n-2) u_\varepsilon^{\frac{n+2}{n-2}}$. Multiplying this equation by u and integrating by parts, we have

$$\int_{\mathbb{R}^n} |Du_\varepsilon|^2\, dx = n(n-2) \int_{\mathbb{R}^n} u_\varepsilon^{\frac{2n}{n-2}}\, dx$$

By direct computation, one can show that $\int_{\mathbb{R}^n} u_\varepsilon^{\frac{2n}{n-2}}\, dx = \frac{\omega_n}{2^n}$. So

$$J(u_\varepsilon) = \frac{\int_{\mathbb{R}^n} |Du_\varepsilon|^2\, dx}{\left(\int_{\mathbb{R}^n} u_\varepsilon^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}}} = \frac{n(n-2)}{4}\, \omega_n^{\frac{2}{n}}$$

By the Sobolev imbedding theorem for $\mathbb{R}^n$ (1.5), for all $u \in H_1$, $\| u \|_{\frac{2n}{n-2}} \leq K(n)\, \| Du \|_2$. So $\mu \geq \left(\frac{1}{K(n)}\right)^2 = \frac{n(n-2)}{4}\, \omega_n^{\frac{2}{n}}$

Thus $\mu = \frac{n(n-2)}{4}\, \omega_n^{\frac{2}{n}}$ by Proposition 1.1 and u_ε is a minimizing function.

Since S^n is equivalent to $\mathbb{R}^n \cup \{\infty\}$ by the stereographic projection, we also have $\mu = \frac{n(n-2)}{4} \omega_n^{\frac{2}{n}}$ for S^n.

Chapter 2 Works of Trudinger and Aubin

§ 2.1 A criterion for solution in terms of μ

In Yamabe's paper, he showed that minimizing functions u_q for functionals J_q exists for $q < \frac{2n}{n-2}$ [Ya,p101]. He hoped that these u_q's have a subsequence uniformly convergent to a non-trivial solution u. Then $u \geq 0$, $u \in C^1$ and satisfies (1.1). By the regularity theorem 1.2, u is smooth and positive. Then the problem is solved. But, as pointed out by Trudinger, Yamabe made an error in his proof that these u_q's are uniformly bounded away from 0. Moreover this fact is not true in general. A counter-example is given on S^n in [Au4,p132].

We first present Trudinger's simplified proof of Yamabe's first claim.

Theorem 2.1 For $2 < q < \frac{2n}{n-2}$, there exists a smooth strictly positive function u_q satisfying $J_q(u_q) = \mu_q$.

Proof. Let $\{u_i\}$ be a minimizing sequence in H_1 such that $\|u_i\|_q = 1$, $u_i > 0$ and $\lim_{i\to\infty} J_q(u_i) = \mu_q$. Then $\{u_i\}$ is a bounded sequence in H_1. Thus $\{u_i\}$ is weakly precompact in H_1 and precompact in L_q by the Kondrakov theorem. There exists a subsequence, also denoted by $\{u_i\}$, and a function $u_q \in H_1$ so that

(i) $u_i \longrightarrow u_q$ in L_q;

(ii) $\quad u_i \longrightarrow u_q$ weakly in H_1;

(iii) $\quad u_i \longrightarrow u_q$ pointwise almost everywhere.

Therefore $\|u_q\|_q = 1$, $u_q \geq 0$ and by Fatou's lemma

$$J_q(u_q) \leq \underline{\lim}_{i\to\infty} J_q(u_i) = \mu_q.$$

This implies $J_q(u_q) = \mu_q$. So u_q is a weak solution of (1.3). By the Sobolev imbedding theorem, $u_q \in H_1 \subset L_{\frac{2n}{n-2}}$. Thus $u_q^{q-2} \in L_r$ where $r = \frac{2n}{(n-2)(q-2)} > \frac{n}{2}$ when $q < \frac{2n}{n-2}$. By Trudinger's lemma u_q is positive and bounded. Smoothness follows from elliptic regularity theory. Q.E.D.

<u>Theorem 2.2</u> The functions u_q obtained from Theorem 2.1 converge almost everywhere to a smooth, nonnegative function u satisfying (1.1).

<u>Proof.</u> Since we have assumed that the volume of M is equal to 1, $\|u_q\|_2 \leq \|u_q\|_q = 1$. Therefore

$$\int_M |Du_q|^2 \, dv \leq \mu_q + \frac{n-2}{4(n-1)} \sup|R| \, \|u_q\|_2^2$$

$$\leq \mu_q + \frac{n-2}{4(n-1)} \sup|R| \tag{2.1}$$

If we take $v = 1$ on M, then $\mu_q \leq J_q(v) = \int_M R \, dv \leq \sup|R|$.

So $\|Du_q\|_2^2 \leq \sup|R| \left(1 + \frac{n-2}{4(n-1)}\right)$. Combining with the fact that $\|u_q\|_2 \leq 1$, we conclude that $\{u_q\}$ is a bounded sequence in H_1. There exists a function u such that

(i) $\quad u_q \longrightarrow u \quad$ weakly in H_1;

(ii) $\quad u_q \longrightarrow u \quad$ strongly in L_2;

(iii) $\quad u_q \longrightarrow u \quad$ pointwise almost everywhere.

For all $v \in H_1$, $2 < q < \frac{2n}{n-2}$, we have

$$\int_M \langle Du_q, Dv\rangle + \frac{n-2}{4(n-1)} Ru_q v \, dv = \mu_q \int_M u_q^{q-1} v \, dv \tag{2.2}$$

By (i), (ii) $\int_M \langle Du_q, Dv\rangle \, dv \longrightarrow \int_M \langle Du, Dv\rangle \, dv$

$$\int_M Rvu_q \, dv \longrightarrow \int_M Ruv \, dv$$

For the term on the right hand side of (2.2), we observe that $u_q \in H_1 \subset L_{\frac{2n}{n-2}}$. So $\| u_q \|_{\frac{2n}{n-2}} \leq$ const. $\| u_q \|_{H_1}$. From (2.1) and the continuity of $q \longmapsto \mu_q$, $\left\{ \| u_q \|_{H_1} \right\}_{2<q<\frac{2n}{n-2}}$ is bounded independent of q. By (iii) and the dominated convergence theorem

$$\int_M vu_q^{q-1} \, dv \longrightarrow \int_M vu^{\frac{n+2}{n-2}} \, dv$$

Therefore from (2.2), u satisfies

$$\int_M \langle Du, Dv \rangle + \frac{n-2}{4(n-1)} Ruv \, dv = \mu \int_M vu^{\frac{n+2}{n-2}} \, dv$$

for all $v \in H_1$. In the other words, u is a weak solution of (1.1). By Theorem 1.2, u is then smooth. Since $u_q > 0$ for all $q < \frac{2n}{n-2}$, $u \geq 0$ in M but it may not be strictly positive. Q.E.D.

By Trudinger's lemma, u is either strictly positive or identically zero. Now our task is to show $\| u \|_2 > 0$. Then u cannot be identically zero. It is Trudinger who first found that if μ is less than some positive constant, then u is non-trivial [Tr2, p269]. But he did not calculate the constant explicitly. In [Au2, p289] Aubin worked out the constant to be $\frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}$ which is the μ for the sphere.

<u>Theorem 2.3</u> If $\mu < \frac{1}{4}n(n-2)\omega_n^{\frac{2}{n}}$, then the function u in Theorem 2.2 is everywhere positive in M.

<u>Proof.</u> By (1.4) and (2.1) for any $\varepsilon > 0$, we have

(2.3) $\| u_q \|_{\frac{2n}{n-2}}^2 \leq (K(n)^2 + \varepsilon)(\mu_q + \sup |R| \, \| u_q \|_2^2) + A(\varepsilon) \, \|u_q\|_2^2$

If $\mu < \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}} = K(n)^{-2}$, there exists an $\varepsilon > 0$ and $\eta > 0$ such that for $\frac{2n}{n-2} - q < \eta$

$$1 - (K(n)^2 + \varepsilon) \mu_q \geq \varepsilon_\circ > 0$$

Since $\| u_q \|_{\frac{2n}{n-2}} \geq \| u_q \|_q = 1$, we rewrite (2.3) as

$$1 \leq (1 - \varepsilon_\circ) + [A(\varepsilon) + (K(n)^2 + \varepsilon) \sup |R|] \| u_q \|_2^2$$

or $\frac{\varepsilon_\circ}{m} \leq \| u_q \|_2^2$ where $m = A(\varepsilon) + (K(n)^2 + \varepsilon) \sup |R|$ which is independent of q. Then we have what we require :

$$\| u \|_2^2 = \lim \| u_q \|_2^2 \geq \frac{\varepsilon_\circ}{m} > 0. \qquad \text{Q.E.D.}$$

Now one may naturally ask a question : Is μ strictly less than $\frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}$ for all compact Riemannian manifolds except those conformally equivalent to S^n? Of course we hope the answer is yes so that the Yamabe problem is solved for such manifolds. We give some results in this direction.

<u>Corollary 2.4</u> Every compact Riemannian manifold of dimension ≥ 3 is conformally equivalent to one which has either constant or positive scalar curvature.

<u>Proof.</u> If $\mu < \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}$, we can apply Theorem 2.3. If $\mu = \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}$, $\mu_q > 0$ for q near $\frac{2n}{n-2}$. By Theorem 2.1 there is a positive function u_q satisfying

$$\frac{4(n-2)}{n-1} \Delta u_q - Ru_q = -\mu_q u_q^{q-1} = -\mu_q u_q^{q-\frac{2n}{n-2}} u_q^{\frac{n+2}{n-2}} .$$

Therefore by a conformal change $\bar{g} = u_q^{\frac{4}{n-2}} g$, $\bar{g}$ has $\mu_q u_q^{q-\frac{2n}{n-2}}$ as its scalar curvature which is positive. Q.E.D.

<u>Corollary 2.5 [Tr2,p271]</u> Every compact Riemannian manifold with dimension ≥ 3 and $\int_M R \, dv \leq 0$ can be conformally deformed to a Riemannian structure of constant scalar curvature.

Proof. $\mu \leqslant J(1) = \int_M R\,dv \leqslant 0$. The corollary follows from Theorem 2.3 immediately.

Corollary 2.6 [Tr2,p271] Every compact Riemannian manifold with dimension ≥ 3 and constant positive scalar curvature may be deformed to one of constant negative scalar curvature.

Corollary 2.6 follows from Aubin's result [Au1,p130] : Every compact Riemannian manifold with dimension ≥ 3 and constant positive curvature admits a metric with scalar curvature R satisfying $\int_M R < 0$.

From Corollary 2.6, if Yamabe problem is true then every Riemannian manifold of dimension ≥ 3 admits a metric of constant negative curvature. But there are topological obstructions for a manifold to have zero or positive scalar curvature metrics. It is known that some compact spin manifolds do not admit a metric of zero scalar curvature. Also the n-torus admits no metric with positive scalar curvature (see [Ka1,p9] for a brief survey).

§ 2.2 Solution for non-conformally flat manifolds of dimension ≥ 6

By Corollary 2.4, we can assume (M,g) has positive scalar curvature. In order to estimate the constant μ , we may try some test functions which have support in a coordinate patch near a point $o \in M$. For any small real number $r > 0$, let

$$G(r) = \frac{1}{r^{n-1}\omega_{n-1}} \int_{\partial B_r(o)} \sqrt{g}\, ds$$

We first show the following two formula for $G(r)$.

(2.4) $$G(r) = 1 - \frac{R}{6n} r^2 + \frac{r^4}{360n(n+2)}[18\Delta R + 8R_{ij}R^{ij} - 3R_{ijkl}R^{ijkl} + 5R^2] + O(r^6)$$

(2.5) $$G(r) = [1 + a(\frac{1 - \cos\alpha r}{\alpha^2})^2](\frac{\sin\alpha r}{\alpha r})^{n-1} + O(r^6)$$

where $a = \frac{1}{90n(n+2)}(18\Delta R - 3S_{ijkl}S^{ijkl} + \frac{4(2n-7)}{n-2} T_{ij}T^{ij})$ and $n(n-1)\alpha^2 = R$. Here R, ΔR, R_{ij}, R_{ijkl}, T_{ij}, S_{ijkl} all mean their values evaluated at o.

Let $(x^1, x^2, \ldots\ldots, x^n)$ be a normal coordinate system centered at o. By careful calculation, we have the Taylor expansion for $\sqrt{g}(x)$, x near o

(2.6) $$\sqrt{g}(x) = 1 - \frac{1}{6} R_{ij}x^ix^j - \frac{1}{12} R_{ij,k}x^ix^jx^k + \frac{x^ix^jx^kx^l}{4!}[\frac{-3}{5} R_{ij,kl} - \frac{2}{15} R_{ipjq}R_{k}{}^{p}{}_{l}{}^{q} + \frac{1}{3} R_{ij}R_{kl}] + O(|x|^5)$$

Proof of (2.4). We just integrate (2.6). We observe by symmetry that

(i) $\int_{\partial B_r(o)} x^ix^j\, ds = \frac{\omega_{n-1}}{n} \delta_{ij}$

(ii) $\int_{\partial B_r(o)} x^ix^jx^k\, ds = 0$;

(iii) $\int_{\partial B_r(o)} (x^i)^4\, ds = \frac{3}{n(n+2)} r^{n+3}\omega_{n-1}$

(iv) $\int_{\partial B_r(o)} (x^i)^2(x^j)^2\, ds = \frac{r^{n+3}}{n(n+2)}\omega_{n-1}$ if $i \neq j$;

(v) $\int_{\partial B_r(o)} x^ix^jx^kx^l\, ds = 0$ if i, j, k, l, cannot be grouped into pairs.

Using these equalities, we obtain from (2.6)

$$G(r) = 1 - \frac{R}{6n}r^2 + \frac{r}{n(n+2)4!}[\frac{6}{5}\Delta R - \frac{2}{15} R_{ij}R^{ij} - \frac{1}{5} R_{ijkl}R^{ijkl} + \frac{2}{3}R_{ij}R^{ij} + \frac{R^2}{3}] + O(r^6)$$
$$= 1 - \frac{R}{6n}r^6 + \frac{r}{n(n+2)360}[18\Delta R + 8R_{ij}R^{ij} - 3R_{ijkl}R^{ijkl} + 5R^2] + O(r^6)$$

Therefore we proved (2.4). Q.E.D.

Since $R_{ijkl}R^{ijkl} = S_{ijkl}S^{ijkl} + \frac{4}{n-2}T_{ij}T^{ij} + \frac{2R^2}{n(n-1)}$,

and $\quad R_{ij}R^{ij} = T_{ij}T^{ij} + \frac{R^2}{n}$,

after substituting these equalities into (2.4), we get another useful formula for $G(r)$:

$$(2.7)\quad G(r) = 1 - \frac{Rr^2}{6n} + \frac{r^4}{360n(n+2)}[18\Delta R - 3S_{ijkl}S^{ijkl} + (8 - \frac{12}{n-2})T_{ij}T^{ij} + (5 + \frac{8}{n} - \frac{6}{n(n-1)})R^2] + O(r^6).$$

Proof of (2.5). We first write down the Taylor expansions for

$$a(\frac{1 - \cos\alpha r}{\alpha^2})^2 \quad \text{and} \quad (\frac{\sin\alpha r}{\alpha r})^{n-1}$$

which are not difficult to check.

$$a(1 - \frac{\cos\alpha r}{\alpha^2})^2 = \frac{ar^4}{4} + O(r^6)$$

$$(\frac{\sin\alpha r}{\alpha r})^{n-1} = 1 - \frac{n-1}{6}\alpha^2 r^2 + \frac{(n-1)(n-2)}{72}\alpha^4 r^4 + \frac{n-1}{120}\alpha^4 r^4 + O(r^6)$$

Therefore if $R = \alpha^2 n(n-1)$,

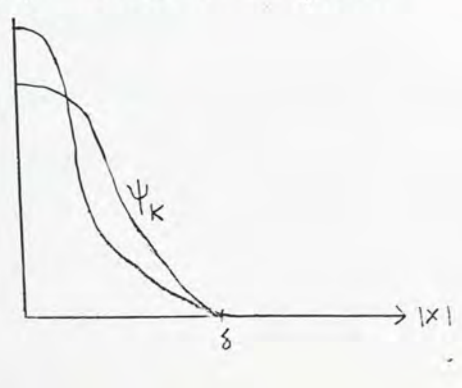
$$[1 + a(\frac{1 - \cos\alpha r}{\alpha^2})^2](\frac{\sin\alpha r}{\alpha r})^{n-1}$$
$$= 1 - \frac{n-1}{6}\alpha^2 r^2 + [\frac{a}{4} + \frac{(n-1)(n-2)}{72}\alpha^4 + \frac{n-1}{120}\alpha^4]r^4 + O(r^6)$$
$$= 1 - \frac{R}{6n}r^2 + \frac{a}{4}r^4 + [\frac{n-2}{72n^2(n-1)} + \frac{1}{120n^2(n-1)}]R^2r^4 + O(r^6)$$
$$= 1 - \frac{R}{6n}r^2 + \frac{a}{4}r^4 + \frac{1}{360n(n+2)}(5 + \frac{8}{n} - \frac{6}{n(n-1)})R^2r^4 + O(r^6)$$

Comparing this equation with (2.7), we obtain (2.5) Q.E.D.

Aubin suggested the following test function to estimate μ for some classes of compact Riemannian manifolds. Let δ be a positive number less than the injectivity radius of o. Take α such that $0<\alpha\delta\leq\pi$. Define for $x\in B_\delta(o)$, $r = d(x,o)$

$$\psi_k(x) = (\frac{1}{k} + \frac{1 - \cos\alpha r}{\alpha^2})^{1-\frac{1}{2}n} - (\frac{1}{k} + \frac{1 - \cos\alpha\delta}{\alpha^2})^{1-\frac{1}{2}n}$$

and $\psi_k(x) = 0$ for $x\in M\backslash B_\delta(o)$.



Lemma 2.7 When $n > 6$

$$J(\psi_k) = \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}[1 + \frac{1}{9n(n-4)(n-6)k^2}(4\,\frac{2n-7}{n-2}\,T_{ij}T^{ij} - 3S_{ijkl}S^{ijkl}) + O(k^{-3})]$$

When $n = 6$

$$J(\psi_k) = \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}} + \frac{\log k\,\omega_{n-1}\,\omega_n^{\frac{2}{n}}\,(n-2)^2}{180n(n+2)k^2\,\omega_n}[\frac{4(2n-7)}{n-2}T_{ij}T^{ij} - 3S_{ijkl}S^{ijkl}] + O(\frac{\log k}{k^3})$$

We now use Lemma 2.7 to give the following theorem and then return to the proof of the lemma.

Theorem 2.8 For those compact Riemannian manifolds with dimension ≥ 6 and not conformally flat, Yamabe's theorem holds.

Proof. Notice that for both $n > 6$ and $n = 6$, the second term in the expression of $J(\psi_k)$ has the factor

$$-\,3\,S_{ijkl}S^{ijkl} + \frac{4(2n-7)}{n-2}T_{ij}T^{ij}$$

We shall show that we can find a conformal metric such that $T_{ij} = 0$ when evaluated at o. Since S_{ijkl} is a conformal invariant, $J(\psi_k) < \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}$ when k is sufficiently large and the theorem is proved.

Let $\bar{g} = e^f g$ be a conformal metric, $\bar{R}_{ij}$,$\bar{R}$ and $\bar{T}_{ij}$ denote the corresponding tensors and function relative to $\bar{g}$. Then

$$\bar{R}_{ij} = R_{ij} - \frac{n-2}{2}D_iD_jf - \frac{\Delta f}{2}g_{ij}\;;$$

$$e^{-f}\bar{R} = R - (n-2)\Delta f;$$

$$\bar{T}_{ij} = T_{ij} - \frac{n-2}{2}D_iD_jf + \frac{n-2}{n}\,f\,g_{ij}.$$

We choose the point $o \in M$ such that $S_{ijkl}(o) \neq 0$ and f is chosen so that f satisfies

$$D_iD_jf(o) = \frac{2}{n-2}T_{ij}(o) \quad \text{for all } i, j.$$

Then $\Delta f(o) = \frac{2}{n-2}g^{ij}T_{ij}(o) = 0$ and $\bar{T}_{ij}(o) = 0.$ Q.E.D.

Proof of Lemma 2.7.

The proof is computational. We have to calculate $\|D\psi_k\|_2^2$, $\int_M R\psi_k^2\,dv$ and $\|\psi_k\|_{\frac{2n}{n-2}}^{-2}$.

Let $I_p^q = \int_0^\infty (1+t)^{-p}\,t^q\,dt$. Using integration by parts, for $p > q+1$, we have

(2.8) $I_{p+1}^q = \frac{p-q-1}{p} I_p^q$ and $I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q$

Moreover when k is sufficiently large ,

(2.9) $\int_0^\gamma (t+\frac{1}{k})^{-p}\,t^q\,dt = \log k + O(1)$ for $p = q+1$,

(2.10) $\int_0^\gamma (t+\frac{1}{k})^{-p}\,t^q\,dt = k^{p-q-1}\left(I_p^q + O(\frac{1}{k})^{p-q-1}\right)$ for $p > q+1$.

We first calculate $\|D\psi_k\|_2^2$, $|D\psi_k| = (\frac{n}{2}-1)(\frac{1}{k}+\frac{1-\cos\alpha r}{\alpha^2})^{-\frac{n}{2}}\frac{\sin\alpha r}{\alpha}$

Let $t = \frac{1-\cos\alpha r}{\alpha^2}$, $\gamma = \frac{1-\cos\alpha\delta}{\alpha^2}$, then $\frac{\sin^2\alpha r}{\alpha^2} = 2t(1-\frac{\alpha^2 t}{2})$

and $dt = \frac{\sin\alpha r}{\alpha}\,dr$. By formula (2.5)

$$
\begin{aligned}
\|D\psi_k\|_2^2 &= (\tfrac{n-2}{2})^2\,\omega_{n-1}\int_0^\delta (t+\tfrac{1}{k})^{-n}(\tfrac{\sin\alpha r}{\alpha})^2\,r^{n-1}\,G(r)\,dr\\
&= (\tfrac{n-2}{2})^2\,\omega_{n-1}\int_0^\delta (t+\tfrac{1}{k})^{-n}[(1+at^2)(\tfrac{\sin\alpha r}{\alpha})^n + O(t^{\frac12 n+3})]\,dt\\
&= (\tfrac{n-2}{2})^2\,\omega_{n-1}\int_0^\delta (t+\tfrac{1}{k})^{-n}(2t)^{\frac12 n}[(1-\tfrac{\alpha^2 t}{2})(1+at^2) + O(t^3)]\,dt\\
&= (\tfrac{n-2}{2})^2\,\omega_{n-1}\int_0^\delta (t+\tfrac{1}{k})^{-n}(2t)^{\frac12 n}[1-\tfrac{n\alpha^2 t}{4}+\tfrac{n(n-2)}{32}\alpha^4 t^2 + at^2 +\\
&\qquad O(t^3)]\,dt
\end{aligned}
$$

By (2.8), (2.10), when $n > 6$

$$
\|D\psi_k\|_2^2 = (\tfrac{n-2}{2})^2\,2^{\frac12 n}\,\omega_{n-1}\,k^{\frac12 n-1}[\,I_n^{\frac12 n} - \tfrac{4\alpha^2}{4k}\,I^{\frac12 n+1} + \tfrac{1}{k^2}(a+\tfrac{n(n-2)}{32}\alpha^4)I_n^{\frac12 n+2} + O(k^{1-\frac12 n})\,]
$$

$$
(2.11)\quad = (\tfrac{n-2}{2})^2\,2^{\frac12 n}\,\omega_{n-1}I_n^{\frac12 n}\,k^{\frac12 n-1}\,[1-\tfrac{n\alpha^2(n+2)}{4k(n-4)} + \tfrac{1}{k}(a+\tfrac{\alpha^4 n(n-2)}{32})\cdot\tfrac{(n+2)(n+4)}{k(n-4)(n-6)} + o(k^{-2})]
$$

Also, when $n = 6$, by (2.9)

$$
(2.12)\quad \|D\psi_k\|_2^2 = (\tfrac{n-2}{2})^2\,2^{\frac12 n}\,\omega_{n-1}\,[k^2\,I_n^{\frac12 n}(1-\tfrac{\alpha^2 n(n+2)}{4k(n-4)}) + (a+\tfrac{\alpha^4 n(n-2)}{32})\log k + O(1)]
$$

Next we calculate $\|\psi_k\|^{-n}_{\frac{2n}{n-2}}$. Let

$$\nu = (\tfrac{1}{k} + \gamma)^{1-\frac12 n} < (1 + \gamma)^{1-\frac12 n}$$

$$\int_M \psi_k^{\frac{2n}{n-2}}\, dv = \omega_{n-1} \int_0^{\gamma} [(t + \tfrac{1}{k})^{1-\frac12 n} - \nu]^{\frac{2n}{n-2}} [(1 + at^2)^{\frac12 n-1}(1 - \tfrac{\alpha^2 t}{2})^{\frac12 n-1} + O(t^{\frac12 n+1})]\, dt$$

Using similar method as in the calculation of $\|D\psi_k\|_2^2$, we have the formula for $\int_M \psi_k^{\frac{2n}{n-2}}\, dv$

$$\int_M \psi_k^{\frac{2n}{n-2}}\, dv = 2^{\frac12 n-1}\, \omega_{n-1}\, k^{\frac12 n}\, I_n^{\frac12 n-1} [1 - \frac{\alpha^2(n-2)n}{4k(n-2)} + \frac{n(n+2)}{k^2}(\frac{a}{(n-2)(n-4)} + \frac{\alpha^4}{32}) + o(\tfrac{1}{k})\ ]$$

Thus for $n \geq 6$, we have

$$(2.13)\qquad \|\psi_k\|^{-2}_{\frac{2n}{n-2}} = [\ 2^{\frac12 n-1}\, \omega_{n-1}\, k^{\frac12 n}\, I_n^{\frac12 n-1}]^{\frac12 n-1} [\ 1 + \frac{\alpha^2(n-2)}{4k} - \frac{n+2}{k^2}(\frac{a}{n-4} + \frac{\alpha^4}{32}(n-2)) + \frac{(n-2)(n-1)}{16k^2} + o(k^{-2})]$$

Finally we compute $\int_M R\psi_k^2\, dv$

$$\int_M R\psi_k^2\, dv = \int_0^{\delta} [(t + \tfrac{1}{k})^{1-\frac12 n} - \nu]^2 \left(\int_{\partial B_r(o)} R\sqrt{g}\ ds\right) r^{n-1}\ dr$$

By (2.6)

$$\int_{\partial B_r(o)} R\sqrt{g}\ ds = \int_{\partial B_r(o)} [\alpha^2 n(n-1) + \frac{\partial R}{\partial x^i}(o)\, x^i + \tfrac12 \frac{\partial^2 R}{\partial x^i \partial x^j}(o)\, x^i x^j + O(r^3)]\, [1 - \tfrac{1}{6} R_{ij}\, x^i x^j + O(r^4)]\ ds$$

$$= \omega_{n-1} [\alpha^2 n(n-1) - \frac{\Delta R}{2n} + \frac{\alpha^4}{6} n(n-1)^2\, r^2] + O(r^4)$$

Put $b = -\dfrac{\Delta R(o)}{n}$.

$$[\alpha^2 n(n-1) + bt](\frac{\sin\alpha r}{\alpha r})^{n-1} = \alpha^2 n(n-1) - (\frac{\Delta R}{2n} + \frac{\alpha^4}{6} n(n-1)^2) r^2 + O(r^4)$$

So we have

$$\int_M R\psi_k^2\, dv = \omega_{n-1} \int_0^{\gamma} (t + \tfrac{1}{k})^{2-n} [(\alpha^2 n(n-1) + bt)(2t)^{\frac12 n-1}(1 - \tfrac{\alpha^2 t}{2})^{\frac12 n-1} + O(t^{1+\frac12 n})]\ dt$$

By (2.9), (2.10), when $n > 6$

$$(2.14)\qquad \int_M R\psi_k^2\ dv = \frac{(n-2)(n-1)}{n(n-4)}\, \omega_{n-1}\, 2^{\frac12 n+1}\, k^{\frac12 n-1}\, I_n^{\frac12 n} [\alpha^2 n(n-1) + \frac{1}{k}(b - \frac{n(n-1)(n-2)}{4}\alpha^4)\frac{n}{n-6} + o(k^{-1})]$$

and when n = 6

$$(2.15) \quad \int_M R\psi_k^2 \, dv = 2^{\frac{1}{2}n-1}\omega_{n-1}\left[4\frac{(n-2)(n-1)^2}{n-4}\alpha^2 I_n^{\frac{1}{2}n} k + (b - \frac{n(n-1)(n-2)}{4}\alpha^4)\log k + O(1)\right].$$

Combining the results, from (2.11), (2.13), (2.14), for n > 6

$$J(\psi_k) = \frac{n(n-2)}{4} 2^{\frac{1}{2}n(n-1)}\omega_{n-1}^{\frac{2}{n}}(I_n^{\frac{1}{2}n-1})^{\frac{1}{2}n}\left[1 + \frac{2b+10(n+2)a}{k^2(n-4)(n-6)} + o(k^{-2})\right]$$

From (2.12), (2.13), (2.14), when n = 6

$$J(\psi_k) = \frac{n(n-2)}{4} 2^{2-\frac{1}{2}n}\omega_{n-1}(I_n^{\frac{1}{2}n-1})^{\frac{1}{2}n}\left[1 + \frac{\log k}{k^2}(a(n-1)(n-2) + \tfrac{1}{2}b) + o(\frac{\log k}{k})\right]$$

To complete the proof, we observe that when n > 6

$$10(n+2)a + 2b = \frac{1}{9n}\left[-3S_{ijkl}S^{ijkl} + \frac{4(2n-7)}{n-2}T_{ij}T^{ij}\right]$$

and when n = 6

$$\tfrac{1}{2}b + a(n-1)(n-2) = \frac{1}{9n(n-2)}\left[-3S_{ijkl}S^{ijkl} + \frac{4(2n-7)}{n-2}T_{ij}T^{ij}\right].$$

Also by direct calculation, it can be shown that

$$2^{n-1}\omega_{n-1} I_n^{\frac{1}{2}n-1} = \omega_n \quad .$$

Q.E.D.

Remarks on the proof of Theorem 2.8.

As pointed out in the Introduction, we see that only local arguments are used. All calculations are carried out in a small neigborhood of a point o in M. Also only the values $R(o)$, $R_{ij}(o)$, $T_{ij}(o)$ and $S_{ijkl}(o)$ are involved in the calculation. Any information about M outside that neighborhood is neglected.

Schoen used different test functions to obtain the same result. A point o is chosen so that $S_{ijkl}(o) \neq 0$. He tried the test function φ_ε which has compact support in a geodesic ball of o and equal to u_ε in a smaller geodesic ball, u_ε being the function defined in section 1.4. Also the metric is conformally deformed so that R, ΔR and T_{ij} vanish at o. By (2.7), we have $G(r) < 1$. Comparing with the results in section 1.4, one expects and indeed gets $J(\varphi_\varepsilon) < \frac{n(n-2)}{4}\omega_n^{\frac{2}{n}}$ as ε is sufficiently small.

Schoen and Aubin used different expressions of $G(r)$ in their calculations. In fact on the sphere of radius $\frac{1}{\alpha}$, $G(r) = (\frac{\sin \alpha r}{\alpha r})^{n-1}$ and on $\mathbb{R}^n$, $G(r) = 1$. Therefore we see that Aubin compared M with S^n while Schoen compared M with $\mathbb{R}^n$.

Chapter 3 Schoen's solution

§ 3.1 Conformally flat case and 3 dimensional case

Let M be a conformally flat Riemannian manifold with metric g_{ij} which is flat in a neighborhood of a fixed point $o \in M$. From Corollary 2.4 we may assume $R \geq 0$ in M. The Green's function of the operator $Lu = \Delta u - \frac{n-2}{4(n-1)}Ru$ exists. In a neighborhood of o where g is flat, we choose a coordinate system $(x^1, x^2, \ldots\ldots, x^n)$ such that $g_{ij} = \delta_{ij}$. Since $R = 0$ near o, $LG = \Delta G = 0$. From section 1.3, we know that $G(x) = G(x,o)$ has the expansion

(3.1) $\qquad G(x) = |x|^{2-n} + A + h(x)$

where $|x| = d(x,o)$, $h(x)$ is a harmonic function satisfying $h(o) = 0$ and A is a constant.

Consider the metric $\overline{g} = G^{\frac{4}{n-2}}g$ in $M\setminus\{o\}$. For x near o, using (3.1)

$$\overline{g} = \sum_{i,j} |x|^{-4}((1 + A|x|^{n-2})\delta_{ij} + O(|x|^{n-1}))dx^i dx^j$$

Let $y = \frac{x}{|x|^2}$ or $x = \frac{y}{|y|^2}$,

Therefore using coordinates $y^1, y^2, \ldots\ldots, y^n$

(3.2) $\qquad \overline{g}_{ij} = (1 + A|y|^{2-n})\delta_{ij} + O(|y|^{1-n})$

Definition 3.1 An oriented Riemannian n-manifold (N,g) is called asymptotically flat if

(i) there exists a compact subset K of N such that $N\setminus K$ consists of finitely many connected components N_1, $N_2, \ldots\ldots, N_r$ with each N_k being diffeomorphic to $\mathbb{R}^n$ minus a ball. N_k is called an end of

N;

(ii) there exists a coordinate system $(x^1, x^2, \ldots, x^n)$ on each N_k such that

$$g_{ij} = \left(1 + \frac{M_k}{2r^{n-2}}\right)^4 \delta_{ij} + p_{ij}$$

where $|p_{ij}| < \frac{k_1}{1+r^{n-1}}$, $|Dp_{ij}| < \frac{k_2}{1+r^n}$, $|D^2 p_{ij}| < \frac{k_3}{1+r^{n+1}}$, with $r = \left(\sum_{i=1}^{n} (x^i)^2\right)^{\frac{1}{2}}$. M_k is a constant and is called the total mass of N_k . $x^1, x^2, \ldots, x^n$ are called asymptotically flat coordinates.

From (3.2), we know that $(M\backslash\{o\}, \bar{g})$ is asymptotically flat with only one end of total mass $\frac{1}{2}A$. By the following theorem of Schoen and Yau, $A \geq 0$ and $A = 0$ if and only if M is conformally equivalent to S^n.

<u>Theorem 3.2</u> Let (N,g) be asymptotically flat. If $R \geq 0$ on N, then the total mass of each end is nonnegative. If, in addition, the total mass M_k of some end N_k is equal to zero, then the metric g is flat. Thus, if N is diffeomorphic to $\mathbb{R}^n$, then N is isometric to $\mathbb{R}^n$.

A proof of this theorem in the 3-dimensional case will be given in Chapter 4.

In our case, suppose M is not conformally equivalent to S^n, $A > 0$. For any small real number ρ_0 , take $\varepsilon_0 < \rho_0$. Let $\psi(x)$ be a piecewise smooth, decreasing function in $|x|$ which satisfies

$$\psi(x) = 1 \text{ for } |x| \leq \rho_0 , \text{ and } \psi(x) = 0 \text{ for } |x| \geq 2\rho_0 .$$

As mentioned in section 1.4, Schoen tried the following test function.

$$\varphi(x) = \begin{cases} u_\varepsilon(x) & \text{for } |x| \leq \rho_0 \\ \varepsilon_0(G(x) - \gamma(x)h(x)) & \text{for } \rho_0 \leq |x| \leq 2\rho_0 \\ \varepsilon_0(G(x)) & \text{for } x \in M\backslash B_{2\rho_0}(o) \end{cases}$$

where $u_\varepsilon(x)$ is the function defined in section 1.4 and $h(x)$ is the harmonic function in (3.1). To require φ to be continuous at $|x| = \rho_0$, we adjust ε so that

$$(3.3) \qquad \varepsilon_0(\rho_0^{2-n} + A) = \left(\frac{\varepsilon}{\varepsilon^2 + \rho_0^2}\right)^{\frac{1}{2}(n-2)}$$

Notice that $\varepsilon_0^2 \approx \varepsilon^{n-2}$ for fixed small ρ_0.

To compute $\int_M |D\varphi|^2 + R\varphi^2 \, dv$ we divide M into two parts : $B_{\rho_0}(o)$ and $M\backslash B_{\rho_0}(o)$. We take ρ_0 small enough such that g is flat in $B_{2\rho_0}(o)$. dx and ds will denote the volume element on $\mathbb{R}^n$ and S^{n-1} with standard metric. dv denotes the volume element of g. In $B_{\rho_0}(o)$ g is flat, dv = dx. So

$$\int_{B_{\rho_0}(o)} |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 \, dv = \int_{B_{\rho_0}(o)} |D\varphi|^2 \, dx \;.$$

Without assuming that g is flat, we have $\int_{B_{\rho_0}(o)} |D\varphi|^2 \, dx$

$$= \int_{B_{\rho_0}(o)} -\varphi\Delta\varphi \, dx + \int_{\partial B_{\rho_0}(o)} \varphi \frac{\partial\varphi}{\partial r} \, ds$$

$$= \int_{B_{\rho_0}(o)} -u_\varepsilon \Delta u_\varepsilon \, dx + \int_{\partial B_{\rho_0}(o)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \, ds$$

$$= n(n-2) \int_{B_{\rho_0}(o)} u_\varepsilon^{\frac{2n}{n-2}} \, dx + \int_{\partial B_{\rho_0}(o)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \, ds$$

$$= n(n-2)\left(\int_{B_{\rho_0}(o)} u_\varepsilon^{\frac{2n}{n-2}} \, dx\right)^{\frac{2}{n}} \left(\int_{B_{\rho_0}(o)} u_\varepsilon^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}} + \int_{\partial B_{\rho_0}(o)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \, ds$$

From section 1.5 we know that

$$\int_{B_{\rho_0}(o)} u_\varepsilon^{\frac{2n}{n-2}} \, dx \leq \int_{\mathbb{R}^n} u_\varepsilon^{\frac{2n}{n-2}} \, dx = 2^{-n}\omega_n$$

Therefore

$$(3.4) \quad \int_{B_{\rho_0}(o)} |D\varphi|^2 \, dx \leq \frac{n(n-2)}{4} \omega_n^{\frac{2}{n}} \left(\int_{B_{\rho_0}(o)} \varphi^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}} + \int_{\partial B_{\rho_0}(o)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \, ds$$

In $M\backslash B_{\rho_0}(o)$ since LG = 0, we have

$$\int_{M\backslash B_{\rho_0}(o)} \left(|D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2\right) dv$$

$$= \varepsilon_0^2 \int_{M\backslash B_{\rho_0}(o)} |DG|^2 + \frac{n-2}{4(n-1)} RG^2 \, dv \; +$$

$$\varepsilon_0^2 \int_{B_{2\rho_0}(o)\backslash B_{\rho_0}(o)} [|D\psi h|^2 - 2\langle DG, D(\psi h)\rangle]\, dx$$

$$= - \varepsilon_0^2 \int_{\partial B_{\rho_0}(o)} G \frac{\partial G}{\partial r} ds + \varepsilon_0^2 \int_{B_{2\rho_0}(o)\backslash B_{\rho_0}(o)} (|D\psi h| - 2\langle DG, D\psi h\rangle)\, dx$$

By the gradient estimates for harmonic functions [GT, p40] $|Dh| \le$ const. and $|DG| = O(|x|^{1-n})$. Then

$$|D\psi h| \le |hD\psi| + |\psi Dh| \le \text{const.}$$

So
$$\int_{B_{2\rho_0}(o)\; B_{\rho_0}(o)} |D\psi\alpha|^2 + 2|DG|\,|D(\psi h)|\, dx$$
$$\le c_1\left(\int_{\rho_0}^{2\rho_0} r^{n-1}\, dr + \int_{\rho_0}^{2\rho_0} r^{1-n}\, r^{n-1}\, dr \right) \le c_2 \rho_0$$

Hence we obtain

(3.5) $$\int_{M\backslash B_{\rho_0}(o)} |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2\, dv \le -\varepsilon_0^2 \int_{\partial B_{\rho_0}(o)} G \frac{\partial G}{\partial r} ds + c_2 \rho_0 \varepsilon_0^2$$

Combining with the results (3.4), (3.5) we have

(3.6) $$\int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2\, dv \le \frac{n(n-2)}{4} \omega_n^{\frac{2}{n}} \left(\int_{B_{\rho_0}(o)} \varphi^{\frac{2n}{n-2}}\, dv \right)^{\frac{1}{2}(n-2)}$$
$$+ \int_{\partial B_{\rho_0}(o)} \left(u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G \frac{\partial G}{\partial r} \right) ds + c_2 \rho_0 \varepsilon_0^2$$

For $|x| = \rho_0$, we observe that u_ε, $\varepsilon_0 G < 2\varepsilon_0 \rho_0^{2-n}$. So

(3.7) $$u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G \frac{\partial G}{\partial r} \le 2\varepsilon_0 \rho_0^{2-n} \left(\frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0 G \frac{\partial G}{\partial r} \right)$$

By (3.3) $$\frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0 \frac{\partial G}{\partial r} = -(n-2)\left[\left(\frac{\varepsilon}{\varepsilon^2 + \rho_0^2} \right)^{\frac{n-4}{2}} \frac{\varepsilon \rho_0}{(\varepsilon^2 + \rho_0^2)^2} - \varepsilon_0 \rho_0^{1-n} \right] + \left(\frac{\partial}{\partial r} h\right)\varepsilon_0$$
$$\le -(n-2)\frac{1}{\rho_0}\left[\left(\frac{\varepsilon}{\varepsilon^2 + \rho_0^2} \right)^{\frac{n-2}{2}} \frac{\rho_0^2}{\varepsilon^2 + \rho_0^2} - \varepsilon_0 \rho_0^{2-n} \right] + C_3 \varepsilon_0$$
$$\doteq -(n-2)\varepsilon_0 \frac{1}{\rho_0}\left[(\rho_0^{2-n} + A) \frac{1}{1 + (\frac{\varepsilon}{\rho_0})^2} - \rho_0^{2-n} \right] + C_3 \varepsilon_0$$

Since $$\frac{1}{1 + (\frac{\varepsilon}{\rho_0})^2} = 1 - \left(\frac{\varepsilon}{\rho_0}\right)^2 + \left(\frac{\varepsilon}{\rho_0}\right)^4 - \left(\frac{\varepsilon}{\rho_0}\right)^6 + \cdots$$
$$\ge 1 - \left(\frac{\varepsilon}{\rho_0}\right)^2$$

$$\frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0 \frac{\partial G}{\partial r} \le -(n-2)\frac{\varepsilon_0}{\rho_0}\left[\rho_0^{2-n} + A - \left(\frac{\varepsilon}{\rho}\right)^2 \rho_0^{2-n} - \frac{\varepsilon^2}{\rho_0^2} A - \rho_0^{2-n} \right] + C_3 \varepsilon_0$$

(3.8) $$\le -(n-2)\frac{\varepsilon_0}{\rho_0} A + \frac{C_4 \varepsilon^2}{\rho_0^{n+1}} + C_3 \varepsilon_0$$

From (3.7), (3.8) $$\int_{\partial B_{\rho_0}(o)} \left(u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G \frac{\partial G}{\partial r} \right) ds$$
$$\le 2\varepsilon_0 \rho_0^{2-n} \left(-(n-2)\frac{\varepsilon_0}{\rho_0} A + \frac{C_4 \varepsilon^2}{\rho_0^{n+1}} + C_3 \varepsilon_0 \right) \rho_0^{n-1} \omega_{n-1}$$

$$= -2(n-2)\,\omega_{n-1}\,\varepsilon_0^2 A + \frac{C_5\,\varepsilon^2\varepsilon_0^2}{\rho_0^n} + C_6\,\varepsilon_0^2\rho_0$$

Now we obtain from (3.6) that

(3.9) $$\int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2\, dv$$

$$\leq \frac{n(n-2)}{4}\,\omega_n^{\frac{2}{n}} \left(\int_{B_{\rho_0}(0)} \varphi^{\frac{2n}{n-2}}\, dv\right)^{\frac{n-2}{n}} - 2(n-2)\omega_{n-1} A\,\varepsilon_0^2 + C_5\frac{\varepsilon^2\varepsilon_0^2}{\rho_0^n} + C_6\varepsilon_0^2\rho_0$$

We first fix ρ_0 to be a small number. Since $\varepsilon \approx \varepsilon_0^{\frac{2}{n-2}}$, we take ε_0 to be very small relative to ρ_0. Then the last two terms can be absorbed by the term $-2(n-2)\omega_{n-1} A\,\varepsilon_0^2$. So (3.9) implies

$$\int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2\, dv < \frac{n(n-2)}{4}\,\omega_n^{\frac{2}{n}} \left(\int_M \varphi^{\frac{2n}{n-2}}\, dv\right)^{\frac{n-2}{n}}$$

By Theorem 2.3 and Corollary 2.4 we have the following result.

<u>Theorem 3.3</u> If M is a compact Riemannian manifold of dimension ≥ 3 , which is conformally flat in some open set and is not conformally diffeomorphic to S^n , then it can be conformally deformed to a Riemannian structure of constant scalar curvature.

<u>In fact we have proved the following statement</u> :

For any compact Riemannian manifold M, of dimension $\geq$ 3, with positive scalar curvature, if the Green's function of the operator L has the expression (3.1) near a point $o \in M$, then equations (3.4), (3.5) hold.

In general, the Green's function does not have the expression (3.1). But when n=3, G has the form

(3.10) $\quad G = \dfrac{1}{|x|} + A + h,\qquad$ where

A and h have the properties as stated in (3.1). Using (3.10), we shall solve the 3-dimensional case of the Yamabe problem.

Theorem 3.4 If M is a compact 3-manifold which is conformally different from S^3, then M is conformally equivalent to a Riemannian structure of constant scalar curvature.

Proof. By the asymptotic formula (2.6) of g, there exists a constant c such that

(3.11) $(1 - c|x|^2)dx \leqslant dv = \sqrt{g}\, dx \leqslant (1 + c|x|^2)dx$

We shall show that

(3.12) $\int_{B_{\rho_0}(o)} |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 \, dv \leq \int_{B_{\rho_0}(o)} |D\varphi|^2 \, dx + c_7 \, \varepsilon \rho_0$

(3.13) $\int_{B_{\rho_0}(o)} \varphi^{\frac{2n}{n-2}} \, dx \leq \int_{B_{\rho_0}(o)} \varphi^{\frac{2n}{n-2}} dv + c_8 \, \varepsilon^2 \rho_0$

By (3.4), (3.5), (3.12), (3.13) when ρ_0 and ε are small enough, we obtain $\int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 dv < \frac{n(n-2)}{4} \omega_n^{\frac{2}{n}} \int_M \varphi^{\frac{2n}{n-2}} dv$ and Theorem 3.4 is proved. Q.E.D.

Proof of (3.12) and (3.13) : By (3.11)

$$\int_{B_{\rho_0}(0)} u_\varepsilon^{\frac{2n}{n-2}} dx \leqslant \int_{B_{\rho_0}(0)} u_\varepsilon^{\frac{2n}{n-2}} dv + c \int_{B_{\rho_0}(0)} \left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^3 |x|^2 dx$$

$$\leqslant \int_{B_{\rho_0}(0)} u_\varepsilon^{\frac{2n}{n-2}} dv + c \int_{B_{\rho_0}(0)} \varepsilon \left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^2 dx$$

$$\leqslant \int_{B_{\rho_0}(0)} u_\varepsilon^{\frac{2n}{n-2}} dv + c\varepsilon^2 \rho_0$$

$$\int_{B_{\rho_0}(0)} \frac{1}{8} R\varphi^2 dv \leqslant \frac{1}{8} \sup_M |R| \int_{B_{\rho_0}(0)} \frac{\varepsilon}{\varepsilon^2+|x|^2} (1+c|x|^2) dx$$

$$\leqslant C \varepsilon \rho_0$$

$$\int_{B_{\rho_0}(0)} |Du_\varepsilon|^2 dv \leqslant \int_{B_{\rho_0}(0)} |Du_\varepsilon|^2 dx + \int_{B_{\rho_0}(0)} \left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)\left(\frac{|x|}{\varepsilon^2+|x|^2}\right)^2 |x|^2 dx$$

$$\leqslant \int_{B_{\rho_0}(0)} |Du_\varepsilon|^2 dx + C\varepsilon\rho_0$$

Q.E.D.

3.2 4 and 5 dimensional case

Let M be a compact Riemannian manifold with positive scalar curvaure. We first claim that we can always conformally deform M such that the scalar curvature $R = 0$ in a neighborhood of a point o in M.

Proof of the claim. Let u be a non-trivial solution of the differential equation $\Delta u - \frac{n-2}{4(n-1)}Ru = 0$ in a neighborhood of o. We can assume $u(o) > 0$. Let φ be a positive function which is equal to u in a small neighborhood of o. Then the metric $\bar{g} = \varphi^{\frac{4}{n-2}}g$ has zero scalar curvature near o. Q.E.D.

In the rest of this chapter, we assume M has 0 scalar curvature in a neighborhood of o, but not necessary positive in M. By Proposition 1.4, the Green's function G of the operator L exists. Let $(x^1, x^2, \ldots\ldots, x^n)$ be a normal coordinate system centered at o. (r, θ) is a corresponding polar coordinates. we write $g = dr^2 + r^2 h_r$

where h_r is a metric on S^{n-1}. Let h_o be the standard metric on S^{n-1}, ρ be a positive small real number and $\zeta(r)$ be a decreasing function satisfying

$$(3.14)\quad \begin{cases} \zeta(r) = 1 & \text{for } r \leq \rho \\ \zeta(r) = 0 & \text{for } r \geq 2\rho \\ |\zeta'(r)| \leq \frac{C_1}{\rho} \text{ and } |\zeta''(r)| \leq \frac{C_2}{\rho^2} & \text{for all } r. \end{cases}$$

Define a family of metric

$$(3.15)\quad \begin{cases} {}^{\rho}g = dr^2 + r^2(\zeta(r)h_o + (1 - \zeta(r))h_r) & \text{in } B_{2\rho_o}(o) \\ {}^{\rho}g = g & \text{outside } B_{2\rho_o}(o) \end{cases}$$

${}^{\rho}g$'s are uniformly equivalent to g. By (3.14), the first and second derivatives of ${}^{\rho}g$ are uniformly bounded. It implies that R_ρ, the scalar curvature of ${}^{\rho}g$ are also uniformly bounded. Let D_ρ and Δ_ρ be the covariant differentiation and Laplacian relative to metric ${}^{\rho}g$. dv_ρ is the volume element.

Lemma 3.5 For all u, $v \in H_1$, as $\rho \longrightarrow 0$

$$\int_M \langle D_\rho u, D_\rho v\rangle + \frac{n-2}{4(n-1)} R_\rho uv \, dv \longrightarrow \int_M \langle Du, Dv\rangle + \frac{n-2}{4(n-1)} uvR \, dv$$

Proof. Since ${}^{\rho}g$ converges to g uniformly, it is clear that

$$\int_M \langle D_\rho u, D_\rho v\rangle \, dv \longrightarrow \int_M \langle Du, Dv\rangle \, dv \quad \text{as } \rho \longrightarrow 0.$$

By using partition of unity, we may assume u and v have support in a coordinate patch U. For all i, j, k, l

$$\int_U uv\, {}^{\rho}g_{ij,kl} \, dv = \int_U uv\sqrt{{}^{\rho}g}\, {}^{\rho}g_{ij,kl} \, dx$$
$$= \int_U (uv\sqrt{{}^{\rho}g})_{,l}\, {}^{\rho}g_{ij,k} \, dx$$

Similarily $\int_U uv\, g_{ij,kl} \, dv = \int_U (uv\sqrt{g})_{,l}\, g_{ij,k} \, dx$.

Since $\int_U (uv\sqrt{{}^{\rho}g})_{,l}\, {}^{\rho}g_{ij,k} \, dx \longrightarrow \int_U (uv\sqrt{g})_{,l}\, g_{ij,k} \, dx$,

therefore $\int_U uv\, {}^{\rho}g_{ij,kl} \, dx \longrightarrow \int_U uv g_{ij,kl} \, dx$.

Since R_ρ and R can be expressed in terms of ${}^{\rho}g_{ij}$, g_{ij} and their first and second derivatives, we have

$$\int uvR_\rho \, dv \longrightarrow \int uvR \, dv \; .$$ Q.E.D.

Let $L_\rho = \Delta_\rho - \frac{n-2}{4(n-1)} R_\rho$. λ and λ_ρ are the first eigenvalues of L and L_ρ respectively. We shall show that the Green's function G_ρ of L_ρ exists and converges to G, the Green's function of L.

Lemma 3.6 (i) $\lambda_\rho \longrightarrow \lambda$ as $\rho \longrightarrow 0$. Since $\lambda > 0$

therefore $\lambda_\rho > 0$ and G_ρ exists for ρ sufficiently small.

(ii) $G_\rho \longrightarrow G$ uniformly in C^2 norm on compact subsets of $M\setminus\{0\}$

(iii) For $n \geq 4$, $x \in B_{2\rho}(o)$, $\alpha \in (3, 4)$

$$| G_\rho(x) - |x|^{2-n} | \leq c_3 |x|^{\alpha-n}$$
$$| D(G_\rho(x) - |x|^{2-n}) | \leq c_4 |x|^{\alpha-n-1}$$

c_3, c_4 depends on α but not on ρ.

Proof. By Lemma 3.5, (i) follows immediately.

We first prove (iii). Notice that $|x|^{2-n}$ and $|x|^{\alpha-n}$ are radical functions. By (3.5) we have $\frac{\partial}{\partial r}\sqrt{g} = O(|x|)$. So

$$\text{(3.16)}\quad \Delta_\rho(|x|^{2-n}) = \frac{1}{|x|^{n-1}\sqrt{{}^\rho g}} \frac{\partial}{\partial r}(|x|^{n-1}\sqrt{{}^\rho g}\frac{\partial}{\partial r}|x|^{2-n}) \leq c_5 |x|^{2-n}$$
$$\Delta_\rho(|x|^{\alpha-n}) = \frac{1}{|x|^{n-1}\sqrt{{}^\rho g}} \frac{\partial}{\partial r}(|x|^{n-1}\sqrt{{}^\rho g}\frac{\partial}{\partial r}|x|^{\alpha-n})$$
$$\leq (\alpha-n)(\alpha-2)|x|^{\alpha-2-n} + C(n-\alpha)|x|^{\alpha-1-n}$$

Since $3 < \alpha < 4$ and $n \geq 4$, we have

$$\text{(3.17)}\quad \Delta_\rho(|x|^{\alpha-n}) \leq -\varepsilon(\alpha)|x|^{2-n}$$

where $\varepsilon(\alpha) > 0$ and depends on α only. By (3.16), (3.17) and the fact that $\{R_\rho\}$ is uniformly bounded, there exists a constant c_6 sufficiently large so that when r is small enough, $x \in B_r(o)\setminus\{o\}$

$$L_\rho(|x|^{2-n} + c_6|x|^{\alpha-n}) \leq 0$$
$$L_\rho(|x|^{2-n} - c_6|x|^{\alpha-n}) \geq 0$$

By maximum principle, since $L_\rho(G_\rho) = 0$

$$|x|^{2-n} - c_6|x|^{\alpha-n} \leq G_\rho(x) \leq |x|^{2-n} + c_6|x|^{\alpha-n}$$

or $\quad |G_\rho(x) - |x|^{2-n}| \leq c_6|x|^{\alpha-n}$

By the interior gradient estimates for elliptic equations [GT,p40]

$$|D_\rho(G_\rho(x) - |x|^{2-n})| \leq c_7|x|^{\alpha-1-n}$$

Since ${}^\rho g$ and its derivatives uniformly converge to g and its derivatives, we obtain

$$|D(G_\rho(x) - |x|^{2-n})| \leq c_8|x|^{\alpha-1-n}$$

By (iii), G_ρ and its derivatives are uniformly bounded on compact subset in $M\setminus\{o\}$. G_ρ must have a limit G' which has singularity of order n-2 at o. G' satisfies $LG' = 0$ in $M\setminus\{o\}$ and

$$\lim_{x\to o} G'(x)|x|^{n-2} = 1.$$

Therefore G' is equal to G and the convergence of G_ρ to G in C^2 norm on compact subsets follows from the interior Schauder estimate. [GT, p85] Q.E.D.

Theorem 3.7 Let M be a compact Riemannian manifold of dimension 4 or 5. If M is not conformally diffeomorphic to S^n, then M can be conformally deformed to a Riemannian structure of constant scalar curvature.

The proof of Theorem 3.7 consists of two steps.

Step 1. Since g is conformally flat in B (o),

(3.18) $G_\rho(x) = |x|^{2-n} + A_\rho + h_\rho(x)$ in $B_\rho(o)$

with $A_\rho > 0$, $h(o) = 0$. We shall show that for suitable test function φ, equation (3.9) holds with A being replaced by A_ρ. If $\lim_{\rho\to o} A_\rho > 0$, then Theorem 3.7 is proved.

Step2. We prove that $\lim_{\rho\to o} A_\rho > 0$ if M is not conformally equivalent to S^n.

Proof of Step 1. As in Theorem 3.3, we define φ as follows :

For fixed small ρ, $2\rho_\circ < \rho$

$$(3.19)\qquad \varphi = \begin{cases} u_\varepsilon(x) & \text{for } |x| \le \rho_0 \\ \varepsilon_0(G_\rho(x) - \psi(x)h_\rho(x)) & \text{for } \rho_0 \le |x| \le 2\rho_0 \\ \varepsilon_0 G_\rho(x) & \text{for } x \in M\setminus B_{2\rho_0}(o) \end{cases}$$

For $|x| = \rho_0$ we adjust ε such that $\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{\frac{n-2}{2}} = \varepsilon_0(|x|^{2-n} + A_\rho)$

Since $R_\rho(o) = R(o) = 0$, from formula (2.4), we have

$$(3.20)\qquad \int_{\partial B_r(o)} \sqrt{g}\, ds = r^{n-1}\omega_{n-1}(1 + O(|x|^4))$$

$$(3.21)\qquad \int_{\partial B_r(o)} \sqrt{g}\, ds = r^{n-1}\omega_{n-1}(1 + O(|x|^4))$$

Let dv and dv_ρ be the volume elements of g and ρg. It follows from (3.20), (3.21) taht if f is a nonnegative radial function on $B_{2\rho}(o)$, for all $r < 2$

$$(3.22)\qquad \left| \int_{B_r(o)} f\, dv - \int_{B_r(o)} f\, dv \right| \le k_1 \int_{B_r(o)} f(x)|x|^4\, dx$$

From now on, we use k_1, k_2,.... etc to denote the constants independent of ρ. Otherwise, we use c_1, c_2,... to denote the constants that may depend on ρ. We shall show that the difference of $\int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2\, dv$ and $\int_M |D_\rho\varphi|^2 + \frac{n-2}{4(n-1)} R_\rho\varphi^2\, dv_\rho$ is very small.

Notice that φ and $|D\varphi| = |\frac{\partial}{\partial r}\varphi|$ are radial functions inside $B_{\rho_0}(o)$. Therefore by (3.21), we obtain

$$(3.23)\qquad \int_{B_{\rho_0}(o)} |D\varphi|^2\, dv \le \int_{B_{\rho_0}(o)} |D_\rho\varphi|^2\, dv_\rho + k_1 \int_{B_{\rho_0}(o)} |D\varphi|^2|x|^4\, dx$$

$$\int_{B_{\rho_0}(o)} |D\varphi|^2|x|^4\, dx = (n-2)^2\varepsilon^{n-2}\int_{B_{\rho_0}(o)} |x|^6 \frac{1}{(\varepsilon^2+|x|^2)^n}\, dx$$

$$\le \frac{(n-2)^2}{6-n}\omega_{n-1}\varepsilon^{n-2}\rho_0^{6-n}$$

Since $n \in \{4,5\}$, $\varepsilon^{n-2} \approx \varepsilon_0^2$, from (3.23) we have

$$(3.24)\qquad \int_{B_{\rho_0}(o)} |D\varphi|^2\, dv \le \int_{B_{\rho_0}(o)} |D_\rho\varphi|^2\, dv_\rho + k_2\varepsilon_0^2\rho_0$$

From (3.18) $\qquad DG_\rho(x) = \frac{\partial}{\partial r}(|x|^{2-n}) + D(h_\rho(x))$

Therefore for $\rho_0 \le |x| \le 2\rho_0$, by (3.19)

$$|D\varphi|^2 = \varepsilon_0^2\left(\frac{\partial}{\partial r}(|x|^{2-n} - D(1-\psi)h_\rho\right)^2$$

$$|D_\rho\varphi|^2 = \varepsilon_o^2(\frac{\partial}{\partial r}(|x|^{2-n}) - D_\rho(1-\psi)h)^2$$

$\frac{\partial}{\partial r}|x|^{2-n}$ is a radial function. $|D(1-\psi)h_\rho| \leq c_9$, $|D_\rho(1-\psi)h| \leq c_{10}$.

It follows from (3.22) that

$$\int_{B_{2\rho_o}(o)\backslash B_{\rho_o}(o)} |D\varphi|^2 \, dv \leq (n-2)^2\varepsilon_o^2\int_{B_{2\rho_o}(o)\backslash B_{\rho_o}(o)} |x|^{2-2n}(1+k_1|x|^4) \, dx$$
$$+ c_{11}\int_{B_{2\rho_o}(o)\backslash B_{\rho_o}(o)} |x|^{1-n} \, dx$$
$$\leq \int_{B_{2\rho_o}(o)\backslash B_{\rho_o}(o)} |D_\rho\varphi|^2 \, dv_\rho +$$
$$\int_{B_{2\rho_o}(o)\backslash B_{\rho_o}(o)} (n-2)^2|x|^{6-2n} + c_{12}|x|^{1-n} \, dx$$

(3.25)
$$\leq \int_{B_{2\rho_o}(o)\backslash B_{\rho_o}(o)} |D_\rho\varphi|^2 \, dv_\rho + c_{13}\rho_o\varepsilon_o^2$$

In $B_{2\rho}(o)\backslash B_\rho(o)$, ρg is not flat, R_ρ is not zero, the calculation will be much delicate. Moreover we require all constants to be independent of ρ.

$$(\frac{\partial}{\partial r}G_\rho)^2 = [\frac{\partial}{\partial r}(G_\rho - |x|^{2-n}) + \frac{\partial}{\partial r}|x|^{2-n}]^2$$
$$= (\frac{\partial}{\partial r}(G_\rho - |x|^{2-n})^2 + 2(2-n)|x|^{1-n}\frac{\partial}{\partial r}(G_\rho - |x|^{2-n}) + (2-n)^2|x|^{2-2n}$$

Let $\hat{D}$ denote the spherial gradient.

$$|DG_\rho|^2 = (\frac{\partial}{\partial r}G_\rho)^2 + |\hat{D}G_\rho|^2$$
$$= (\frac{\partial}{\partial r}G_\rho)^2 + |\hat{D}(G_\rho - |x|^{2-n})|^2$$
$$= (2-n)^2|x|^{2-2n} + |D(G_\rho - |x|^{2-n})|^2 + 2(2-n)|x|^{1-n}\frac{\partial}{\partial r}(G_\rho - |x|^{2-n})$$

We have a similar formula for $|D_\rho G_\rho|$. By (3.22) since $|x|^{2-2n}$ is a radial function,

$$\int_{B_{2\rho}(o)\backslash B_\rho(o)} |DG_\rho|^2 \, dv \leq \int_{B_{2\rho}(o)\backslash B_\rho(o)} (2-n)^2|x|^{2-2n} + (2-n)^2|x|^{6-2n} \, dx$$
$$+ \int_{B_{2\rho}(o)\backslash B_\rho(o)} |D(G_\rho - |x|^{2-n})|^2 + 2(2-n)|x|^{1-n}\frac{\partial}{\partial r}(G - |x|^{2-n}) \, dv$$

From formula (2.6), for any function f (not necessary radial)

$$\int_{B_{2\rho}(o)} f \, dv \leq \int_{B_{2\rho}(o)} f \, dv_\rho + k_3\int_{B_{2\rho}(o)} f|x|^2 \, dx$$

So we have
$$\int_{B_{2\rho}(o)\backslash B_\rho(o)} |DG_\rho|^2 \, dv$$
$$\leq \int_{B_{2\rho}(o)\backslash B_\rho(o)} |D_\rho G_\rho|^2 \, dv_\rho + (2-n)^2\int_{B_{2\rho}(o)\backslash B_\rho(o)} |x|^{6-2n} \, dx$$
$$+ k_3\int_{B_{2\rho}(o)\backslash B_\rho(o)} |D(G_\rho - |x|^{2-n})|^2|x|^2 +$$

$$|2(2-n)|x|^{1-n}\frac{\partial}{\partial 2}(G_\rho - |x|^{2-n})|\ |x|^2\, dx$$

By Lemma 3.6 (iii), $|D(G_\rho - |x|^{2-n})| \leq |x|^{\alpha-n-1}$, we have

$$\int_{B_{2\rho}(0)\setminus B_\rho(0)} |DG_\rho|^2\, dv \leq \int_{B_{2\rho}(0)\setminus B_\rho(0)} |D_\rho G_\rho|^2\, dv_\rho + k_4(\rho^{6-n} + \rho^{2\alpha-n} + \rho^{\alpha+2-n})$$

Take = 3.5, then

(3.26) $$\int_{B_{2\rho}(0)\setminus B_\rho(0)} |D\varphi|^2\, dv \leq \int_{B_{2\rho}(0)\setminus B_\rho(0)} |D_\rho\varphi|^2\, dv_\rho + k_5\, \rho^{\frac{1}{2}}\, \varepsilon_0^{\,2}$$

In $B_{2\rho}(o)$ $R = 0$. But $R_\rho \neq 0$. We have to show that $\int_{B_{2\rho}(0)} R_\rho \varphi^2\, dv_\rho$ is very small. Using normal coordinates, we have $g_{ij} = \delta_{ij} + O(|x|^2)$, $\Gamma^i_{jk} = O(|x|)$

Since ${}^\rho g \longrightarrow g$ and ${}^\rho\Gamma^i_{jk} \longrightarrow \Gamma^i_{jk}$ as $\rho \longrightarrow 0$

$${}^\rho g_{ij} = \delta_{ij} + O(|x|^2) \quad , \quad {}^\rho\Gamma^i_{jk} = O(|x|)$$

We also have

$${}^\rho R^i{}_{jkl} = \tfrac{1}{2}\,{}^\rho g^{ih}({}^\rho g_{hk,jl} - {}^\rho g_{hl,jk} - {}^\rho g_{jk,hl} + {}^\rho g_{jl,hk}) + O(|x|^2)$$

Hence $${}^\rho R = \sum_{i,j} ({}^\rho g_{ij,ij} - {}^\rho g_{ii,jj}) + O(|x|^2)$$

For all $\rho < \delta < 2\rho$, by Stoke's theorem

(3.27) $$\int_{B_\delta(o)} R_\rho\, dv = \int_{\partial B_\delta(o)} \sum_{i,j} ({}^\rho g_{ij,i} - {}^\rho g_{ii,j})\, r_j\, ds + O(\delta^{n+2})$$

Similarily since $R = 0$ in $B_{2\rho}(o)$

(3.28) $$0 = \int_{B_\delta(o)} R\, dv = \int_{\partial B_\delta(o)} \sum_{i,j} (g_{ij,i} - g_{ii,j})\, r_j\, ds + O(\delta^{n+2})$$

From (3.15) ${}^\rho g = \zeta(\delta_{ij}) + (1-\zeta)g$. When $|x| = \delta$

$$\begin{aligned}{}^\rho g_{ij,i} - {}^\rho g_{ii,j} &= \zeta'(\delta)(\delta_{ij} - g_{ij})\, r_i - \zeta'(\delta)(\delta_{ii} - g_{ii})\, r_j \\ &\quad + (1-\zeta)(\delta)(g_{ij,i} - g_{ii,j})\end{aligned}$$

Substitute the above equation in (3.27), by (3.28)

$$\begin{aligned}\int_{B_\delta(o)} R_\rho\, dv_\rho = \zeta'(\delta) \int_{\partial B_\delta(o)} &\sum_{i,j} (\delta_{ij} - g_{ij})\, r_i r_j - \\ &\sum_{i,j} (\delta_{ii} - g_{ii})\, r_j r_j\, ds + O(\delta^{n+2})\end{aligned}$$

The vector $Dr = \sum_i r_i \frac{\partial}{\partial x^i}$ is in the radial direction. Since we are using normal coordinates, $\sum_{i,j} g_{ij}\, r_i\, r_j = 1$.

In the Taylor expansion,

$$g_{ii} - \delta_{ii} = \tfrac{1}{2} \sum_{j,k} g_{ii},_{jk}(o)x^j x^k + \sum_{j,k,l} g_{ii},_{jkl}(o)x^j x^k x^l + O(r^4)$$

By symmetry $\int_{\partial B_\delta(o)} x^j x^k \, ds = 0$ when $j \neq k$

and $\int_{\partial B_\delta(o)} x^j x^k x^l \, ds = 0$ for all j, k, l.

$$\int_{\partial B_\delta(o)} R_\rho \, dv_\rho = \zeta'(\delta) \int_{\partial B_\delta(o)} \tfrac{1}{2} \sum_{i,j} g_{ii},_{jj}(o) \, ds + O(\delta^{n+2})$$

Since $R(o) = -3\tfrac{1}{2} \sum_{i,j} g_{ii},_{jj}(o) = 0$, we have

(3.29) $\int_{\partial B_\delta(o)} R_\rho \, dv_\rho = O(\delta^{n+2})$

By Lemma 3.6 (iii), since R_ρ are uniformly bounded

$$\begin{aligned} \left| \int_{B_{2\rho}(o)\setminus B_\rho(o)} R_\rho G_\rho^2 \, dv_\rho \right| &\leq \left| \int_{B_{2\rho}(o)\setminus B_\rho(o)} R_\rho |x|^{4-2n} \, dv_\rho \right| + \\ &\quad \left| k_5 \int_{B_{2\rho}(o)\setminus B_\rho(o)} |x|^{2+\alpha-2n} \, dx \right| \\ &= \left| \int_{B_{2\rho}(o)\setminus B_\rho(o)} R_\rho |x|^{4-2n} \, dv_\rho \right| + k_6 \rho^{2+\alpha-n} \end{aligned}$$

Moreover, using integration by parts and (3.29)

$$\begin{aligned} &\int_{B_{2\rho}(o)\setminus B_\rho(o)} R_\rho |x|^{4-2n} \, dv_\rho \\ &= \int_\rho^{2\rho} r^{4-2n} \left(\frac{d}{dr} \int_{\partial B_\delta(o)} R_\rho \, dv_\rho \right) dr \\ &= 2\rho^{4-2n} \int_{B_{2\rho}(o)} R_\rho \, dv_\rho + (2n-4) \int_\rho^{2\rho} r^{3-2n} \left(\int_{\partial B_r(o)} R_\rho \, dv_\rho \right) dr \\ &= k_7 \rho^{6-n} . \end{aligned}$$

Therefore we have

(3.30) $\int_{B_{2\rho}(o)\setminus B_\rho(o)} R_\rho \varphi^2 \, dv_\rho \leq k_8 \rho^{\frac{1}{2}} \varepsilon_0^2$

Combining the results (3.24), (3.25), (3.26), (3.30), we obtain

$$\begin{aligned} &\int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 \, dv \\ &\leq \int_M |D_\rho \varphi|^2 + \frac{n-2}{4(n-1)} R_\rho \varphi^2 \, dv_\rho + c_{14} \rho_0 \varepsilon_0^2 + k_9 \rho^{\frac{1}{2}} \varepsilon_0^2 \end{aligned}$$

Since ρ_g is locally conformally flat near o, by (3.9)

(3.31) $$\begin{aligned} \int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 dv &\leq \frac{n(n-1)}{4} \omega_n^{\frac{2}{n}} \left(\int_{B_{\rho_0}(o)} \varphi^{\frac{2n}{n-2}} dv_\rho \right)^{\frac{n-2}{n}} \\ &\quad - 2(n-2)\omega_{n-1} A_\rho \varepsilon_0^2 + C_{15} \frac{\varepsilon^2 \varepsilon_0^2}{\rho_0^n} \\ &\quad + C_{16} \rho_0 \varepsilon_0^2 + k_9 \rho^{\frac{1}{2}} \varepsilon_0^2 \end{aligned}$$

By (3.22) $$\begin{aligned} &\int_{B_{\rho_0}(o)} u_\varepsilon^{\frac{2n}{n-2}} dv_\rho \\ &\leq \int_{B_{\rho_0}(o)} u_\varepsilon^{\frac{2n}{n-2}} dv + k_1 \int_{B_{\rho_0}(o)} u_\varepsilon^{\frac{2n}{n-2}} |x|^4 dx \\ &\leq \int_{B_{\rho_0}(o)} u_\varepsilon^{\frac{2n}{n-2}} dv + k_{10} \varepsilon^n \rho_0^{4-n} \end{aligned}$$

so $\quad \left(\int_{B_{\rho_0}(o)} \varphi^{\frac{2n}{n-2}} dv_\rho \right)^{\frac{n-2}{n}} \le \left(\int_{B_{\rho_0}(o)} \varphi^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} + k_{11} \frac{\varepsilon^n}{\rho_0^{n-4}}$

From (3.31) and $\quad \varepsilon_0 \approx \varepsilon^{\frac{n-2}{2}}$

$$\int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 dv \le \frac{n(n-2)}{4} \omega_n^{\frac{2}{n}} \left(\int_M \varphi^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}$$
$$- 2(n-2)\omega_{n-1} A_\rho \varepsilon_0^2 + C_{15} \varepsilon_0^{\frac{2n-2}{n-2}} \rho_0^{-n} + C_{16} \rho_0 \varepsilon_0^2$$
$$+ k_9 \rho^{\frac{1}{2}} \varepsilon_0^2 + k_{12} \varepsilon_0^{\frac{2n}{n-2}} \rho_0^{4-n}$$

If $\lim_{\rho \to 0} A_\rho > 0$, we first fix ρ small then fix ρ_0 and choose ε_0 very small relative to ρ_0 so that all error terms can be absorbed by $A_\rho \varepsilon_0^2$. We obtain what we require

$$\int_M |D\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 \, dv < \frac{n(n-2)}{4} \omega_n^{\frac{2}{n}} \left(\int_M \varphi^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}$$

This ends Step 1. Q.E.D.

Proof of Step2.

Lemma 3.8 If the metric $G^{\frac{4}{n-2}} g$ is not Ricci flat on $M \setminus \{o\}$, then $\lim_{\rho \to 0} A_\rho > 0$.

Proof. Fix a small ρ, let $\bar{g} = {}^\rho g$ and $\bar{G} = G_\rho$. K is a compact subset of $M \setminus \{o\}$ on which $G^{\frac{4}{n-2}} g$ is not Ricci flat. $S = (S_{ij})$ is a tensor with compact support in K. Define a family of metric

$$g^t = \bar{G}^{\frac{4}{n-2}} \bar{g} + tS$$

It is positive definite if $|t| < c$ where c is a constant depending on $\bar{g}$ and S only. g^0 is scalar flat. Let $(R^t_{ij}) = \mathrm{Ric}(g^t)$ and u_t be a solution of

$$(3.32) \quad \begin{cases} \Delta_t u_t - \frac{n-2}{4(n-1)} R^t u_t = 0 & \text{on } M \\ u_t(o) = 1 \end{cases}$$

In fact, if H_t is the normalized Green's function of $\bar{g} + t\bar{G}^{\frac{-4}{n-2}}S$, $u_t = H_t\bar{G}^{-1}$ is a solution. The metric $\bar{g}$ and g^t are flat near o, for $|x|$ small, we have

$$H_t(x) = |x|^{2-n} + \bar{A}_t + O(|x|)$$

$$\bar{G}(x) = |x|^{2-n} + \bar{A} + O(|x|)$$

Hence $u_t(x) = 1 + (\bar{A}_t - \bar{A})|x|^{n-2} + O(|x|^{n-1})$

Integrating the equation (3.32) over $M\backslash B_r(o)$, using Stoke's theorem and letting $r \longrightarrow 0$, we have

(3.33) $\quad 4(n-1)\omega_{n-1}(\bar{A} - \bar{A}_t) = \int_{M\backslash\{o\}} R^t u_t \; dv^t$

where dv^t is the volume element of g^t.

<u>Claim</u> : $\frac{d}{dt}\int_{M\backslash\{o\}} R^t u_t \; dv^t|_{t=0} = -\int_{M\backslash\{o\}} S_{ij}\,(R^o)^{ij}\,dv^o$

Suppose the claim is true. By Lemma 3.6 $G_\rho = \bar{G}$ is close to G in C^2 norm on compact subset of $M\backslash\{o\}$. If ρ is small enough so that $B_{2\rho}(o)\cap K = \phi$, ${}^\rho g = \bar{g} = g$. Then $\bar{G}^{\frac{4}{n-2}}\bar{g}$ is also not Ricci flat in K. Let χ be a smooth nonnegative function with compact support in K. $S = -\chi\,\mathrm{Ric}(\bar{G}^{\frac{4}{n-2}}\bar{g})$. There exists a constant $c_1 > 0$ and independent of ρ such that for ρ small enough

(3.34) $\quad \frac{d}{dt}\int_{M\backslash\{o\}} R^t u_t \; dv^t|_{t=0} \geq c_1 > 0$

By Theorem 3.2, $\bar{A} > 0$ and $\bar{A}_t > 0$ for all t. From (3.33), (3.34), we know that the mapping $t \longmapsto \bar{A} - \bar{A}_t$ is increasing at $t = 0$. Notice that when $t = 0$, $H_t = \bar{G}$, $u_t = 1$ in $M\backslash\{o\}$. So $\bar{A} = \bar{A}_o$. Thus there esists $c_2 > 0$ and independent of ρ so that for some t small enough, $\bar{A} - \bar{A}_t > c_2 > 0$. Hence

$$\bar{A} = A_\rho = \bar{A}_t + (\bar{A} - A_t) \geq c_2 > 0$$

This proves Lemma 3.8. Q.E.D.

To prove the claim, we first show that if g is a metric, $S = (S_{ij})$ is a symmetric tensor, R_t is the scalar curvature of the metric $g^t = g + tS$

$$(3.35) \qquad \frac{d}{dt} R_t|_{t=0} = -\Delta S + D_i D_j S^{ij} - S_{ij}R^{ij}.$$

For all $x_o \in K$, take a normal coordinate system centered at x_o with respect to g^o, say $(x^1, x^2, \ldots, x^n)$.

$$(\Gamma^t)^k_{ij} = \tfrac{1}{2}(g^t)^{kl}(g_{il'j} + g_{jl'i} - g_{ij'l} + tS_{il'j} + tS_{jl'i} - tS_{ij'l})$$

$$(R^t)^i_{jkl} = \frac{\partial}{\partial x^k}(\Gamma^t)^i_{jl} - \frac{\partial}{\partial x^l}(\Gamma^t)^i_{jk} + (\Gamma^t)^h_{jl}(\Gamma^t)^i_{hk} - (\Gamma^t)^h_{jh}(\Gamma^t)^i_{hl}$$

Since $(\Gamma^t)^k_{ij}(x_o)|_{t=0}=0$ and

$$\frac{d}{dt}\left(\frac{\partial}{\partial x^l}(\Gamma^t)^k_{ij}(x_o)\right)|_{t=0} = \tfrac{1}{2}(S_{ik'jl} + S_{jk'il} - S_{ij'kl})(x_o)$$

we have

$$\frac{d}{dt}(R^t)_{jk}(x_o)|_{t=0} = -\tfrac{1}{2}\sum_l (S_{jl'lk} + S_{ll'jk} - S_{jl'lk})(x_o) + \tfrac{1}{2}\sum_l (S_{jl'kl} + S_{kl'jl} - S_{jk'll})(x_o)$$

Therefore $\quad g^{jk}(\frac{d}{dx}(R^t)_{jk}|_{t=0})(x_o) = \sum_{i,j}(S_{ij'ij}(x_o) - S_{ii'jj}(x_o))$

Also $\quad \frac{d}{dt}(g^t)^{ij}|_{t=0} = - g^{ik}g^{lj}S_{kl}$

$$\frac{d}{dt} R_t|_{t=0} = (\frac{d}{dt}(g^t)^{jk}|_{t=0}) R_{jk} + g^{jk}(\frac{d}{dt}(R^t)_{jk}|_{t=0})$$

When evaluated at x_o

$$(\frac{d}{dt} R_t|_{t=0})(x_o) = - (S_{kl}R^{kl})(x_o) + \sum_{i,j} S_{ij'ij}(x_o) - \sum_{i,j} S_{ii'jj}(x_o)$$
$$= -\Delta S(x_o) + D_i D_j S^{ij}(x_o) - (S_{ij}R^{ij})(x_o)$$

So equation (3.35) is proved.

In our case, since $R^o = 0$ and $u_o = 1$ in $M\setminus\{o\}$

$$\frac{d}{dt}\int_{M\setminus\{o\}} R^t u_t \, dv^t| = \int_{M\setminus\{o\}} \frac{d}{dt} R^t|_{t=0} \, dv^o$$
$$= \int_{M\setminus\{o\}} -\Delta S + D_i D_j S^{ij} - S_{ij}(R^o)^{ij} \, dv^o$$

Since S has compact support in $M\setminus\{o\}$, using integration by parts, the first two terms vanish. This proves our claim. Q.E.D.

Since R = 0 in a neighborhood of o, $LG(x) = \Delta G(x) = 0$ for x near o. Therefore $G = |x|^{2-n} + O(|x|^{4-n})$ [Au5,p108]. If $\bar{g} = G^{\frac{4}{n-2}} g$, we have

(3.36) $$\bar{g}_{ij} = \delta_{ij} + O(|y|^{-2})$$

We shall use (3.36) to show that $M\setminus\{o\}$ is isometric to $\mathbb{R}^n$ if $\bar{g}$ is Ricci flat.

Proposition 3.9 Let $(N,\bar{g})$ be a Riemannian manifold which has only one end. Suppose that there exists a coordinate system on this end such that $\bar{g}$ satisfies (3.36). If $\bar{g}$ is Ricci flat, then $(N,\bar{g})$ is isometric to $\mathbb{R}^n$ with the standard metric.

Proof. We assume the asymptotically flat coordinates y^i's can be chosen to be harmonic coordinates. Then

(3.37) $$0 = \Delta y^i = \frac{1}{\sqrt{\bar{g}}}\frac{\partial}{\partial y^k}(\sqrt{\bar{g}}\ \bar{g}^{ki})$$

Hence the Laplacian becomes $\Delta = \bar{g}^{kl}\frac{\partial^2}{\partial y^k \partial y^l}$

The Ricci flat condition then implies

$$|\Delta \bar{g}_{ij}| \le c|D\bar{g}|^2 = O(|y|^{-6})$$

By elliptic theory, $\bar{g}_{ij}$ has the expression

(3.38) $$\bar{g}_{ij}(y) = \delta_{ij} + A_{ij}|y|^{2-n} + O(|y|^{1-n})$$

where (A_{ij}) is a symmetric nXn metrix. By rotating the coordinates, we may assume (A_{ij}) is a diagonal matrix. We shall show that $A_{jj} = 0$ for all j.

From (3.37) and (3.38), we have

$$\frac{1}{2}\sum_i A_{ii} - A_{jj} = 0 \qquad \text{for all } j.$$

It cannot be true unless $A_{ii} = 0$ for all i. Therefore we conclude that $A_{jj} = 0$ for all j and

(3.39) $$\bar{g}_{ij} = \delta_{ij} + O(|y|^{1-n})$$

Since N has only one end, there is a compact set K such that $N\setminus K = \mathbb{R}^n\setminus B_\sigma(o)$. Choose a large number $r > \sigma$ and let v^i be a solution of the Dirihlet problem :

$$\begin{cases} \Delta v^i = 0 & \text{in } B_r(0)\cup K \\ v^i = y^i & \text{on } \partial B_r(0) \end{cases}$$

We may assume y^i is a global harmonic function on N by taking y^i to be v^i outside $B_r(o)$. Possibly y^i's are linearly dependent outside $B_r(o)$. We claim y^i's must be linearly independent in our case.

By the Bochner formula [Wu]

$$\Delta|Dy^i|^2 = 2|D^2y^i|^2 + 2\mathrm{Ric}(Dy^i).$$

By the Ricci flat condition $\Delta|Dy^i|^2 = 2|D^2y^i|^2$. By (3.39)

$$\Delta|Dy^i|^2 = \bar{g}_{ii} = 1 + O(|y|^{1-n}) \quad \text{and} \quad D|Dy^i|^2 = O(|y|^{-n})$$

For any r sufficiently large, by Stoke's theorem

$$2\int_{B_r(o)} |D^2y^i|^2\, dv = \int_{B_r(0)} \Delta|Dy^i|^2\, dv = \int_{\partial B_r(0)} \frac{\partial}{\partial n}|Dy^i|^2\, ds = O(r^{-1})$$

Taking r tends to ∞ then shows that $D^2y^i = 0$ and so Dy^i is a parallel vector field for each i. Define a map

$$(y^1, y^2, \ldots, y^n) : N \longrightarrow \mathbb{R}^n.$$

It is an isometry by the above discussion. Q.E.D.

By Proposition 3.9, $M\setminus\{o\}$ is isometric to $\mathbb{R}^n$. So M is isometric to S^n. This complete the proof of Step 2.

Chapter 4 The positive mass theorem

§ 4.1 Statements of results

In this chapter, we shall prove the Theorem 3.2. But only the 3-dimensional case is presented here.

Let (N,g) be an asymptotically flat 3-manifold. By Definition 3.1, on each end N_k, there exists a coordinate system (x^1, x^2, x^3) such that

(4.1) $$g_{ij} = \left(1 + \frac{M_k}{2r}\right)\delta_{ij} + p_{ij}.$$

where $r = \left(\sum_{i=1}^{3} (x^i)^2\right)^{\frac{1}{2}}$, $|p_{ij}| \leq \frac{k_1}{1+r^2}$, $|Dp_{ij}| \leq \frac{k_2}{1+r^3}$, $|D^2p_{ij}| \leq \frac{k_3}{1+r^4}$. Then $\Gamma^k_{ij} = O(\frac{1}{r^2})$ and $R_{ijkl} = O(\frac{1}{r^3})$. Moreover g is uniformly equivalent to the Euclidean metric on N_k. Now we state our theorems.

Theorem A Let (N,g) be an asymptotically flat 3-manifold. If $R \geq 0$ on N, then the total mass of each end is nonnegative.

Theorem B Let (N,g) be an asymptotically flat 3-manifold. Suppose that some end N_k has zero total mass. If $R \geq 0$ on N, then g is flat.

In the proof of the Theorem B, we need one more assumption on g :

(4.3) $$|D^3p_{ij}| + |D^4p_{ij}| + |D^5p_{ij}| \leq \frac{k_4}{1 + r^5}$$

§ 4.2 Proof of Theorem A

In this section, we shall work on each fixed N_k. Let N_k be diffeomorphic to $R^3 \setminus B_\sigma(0)$, with coordinates x^1, x^2, x^3 as in section 4.1. We assume that the total mass M_k of N_k is negative and try to derive a contradiction. For simplicity, we denote M_k by M.

The proof consists of three steps. In Step 1, we conformally change the metric to another asymptotically flat metric with scalar curvature $R \geq 0$ on N and $R > 0$ outside a compact subset of N_k. In Step 2, we construct a 'nice' minimal surface S properly imbedded in N. Let K be the Gauss curvature of S. We show both $\int_S K \geq 0$ and $\int_S K < 0$ in Step 3 which gives a contradiction.

Step 1. First we take a function ζ on $(0,\infty)$ such that

$\zeta' \geq 0$ and $\zeta'' \leq 0$ everywhere and

$$\zeta(t) = \begin{cases} t & \text{for } t < -\dfrac{M}{8\sigma} \\ -\dfrac{3M}{16\sigma} & \text{for } t > -\dfrac{M}{4\sigma} \end{cases}$$

Then we take $u = \zeta(-\frac{M}{4r})$ on N_k and $u = 1 - \frac{3M}{16\sigma}$ on $N \setminus N_k$.

Finally we define $\bar{g} = u^4 g$.

On $\mathbb{R}^3 \setminus B_{2\sigma}(o)$, $\bar{g}_{ij} = (1 + \frac{M}{4r})^4 \delta_{ij} + \bar{p}_{ij}$ and $\bar{g}_{ij}$ satisfies the condition of an asymptotically flat metric with total mass equal to $\frac{1}{2}M < 0$. It can be shown that $\Delta u \leq 0$ on N and $\Delta u < 0$ outside a compact subset of N_k. The scalar curvature $\bar{R}$ is given by the formula

$$\bar{R} = u^{-5}(-8\Delta u + Ru)$$

Thus $\bar{R} \geq 0$ on N and $\bar{R} > 0$ outside a compact subset of N_k. Q.E.D.

In Step 2, 3 we assume $R \geq 0$ on N and $R > 0$ on $N_k \approx \mathbb{R}^3 \backslash B_\sigma(o)$.

Step 2. We shall construct a complete area minimizing surface S properly imbedded in N so that $S \cap (N \backslash N_k)$ is compact and $S \cap N_k$ lies between two parallel planes in $\mathbb{R}^3 \backslash B_\sigma(0)$.

For any $t > \sigma$, let C_t be the circle of Euclidean radius t centered at 0 in x^1x^2-plane. We assume there exists a smooth, compact, imbedded, oriented surface S_t of least g-area among all competing surfaces regardless of topological type having boundary C_t. Using the fact that N_k has negative total mass, we obtain:

(i) there is a compact subset $K \subset N$ such that $S_t \cap (N \backslash N_k) \subset K$, for all $t > \sigma$;

(ii) there exists a positive number h such that $S_t \cap N_k \subset E_h$, where $E_h = \{x \in \mathbb{R}^3 : |x^3| \leq h\}$.

For all $x \in N$, let $B_r(x)$ denote the geodesic ball of radius r about x. By a local interior regularity estimate for area minimizing surface, for each $x_o \in S_t$ there is a number $r_o > 0$ with $B_{r_o}(x_o) \cap \partial S_t = \phi$ such that $S_t \cap B_{r_o}(x_o)$ can be written as the graph of a smooth function f_t over the tangent plane $T_{x_o}(S_t)$ in a normal coordinate system on $B_{r_o}(x_o)$. The derivatives of f_t are bounded up to order 3 and the bound is independent of t. Therefore there exists a subsequence, also denoted by S_t, and a complete area minimizing surface S such that S_t converges to S in C^2 norm. S must have the required properties.

Step 3. (i) Proof of $\int_S K > 0$.

Let e_1, e_2, e_3 be orthonormal vector fields defined locally on N such that e_1, e_2 are tangent to S and e_3 is a unit normal of S. Let K_{ij} denote the sectional curvature of the plane $\langle e_i, e_j \rangle$. The Ricci tensor can be written as

$\mathrm{Ric}(e_i) = \sum_{j=1}^{3} K_{ij}$ where we put $K_{ii} = 0$. The scalar curvature R is then given by $R = K_{12} + K_{13} + K_{23}$. $A = (h_{ij})$ denotes the second fundamental form of S, $h_{ij} = \langle D_{e_i} e_3, e_j \rangle$. Since S is minimal, Trace $A = h_{11} + h_{22} = 0$. Therefore

$$R - K + \tfrac{1}{2}\|A\|^2 = \mathrm{Ric}(e_i) + \|A\|^2 \text{ , where } \|A\|^2 = \sum_{i,j} h_{ij}^2$$

The second variation inequality states that for any C^2 function f with compact support on S

$$\int_S f(\triangle f + \mathrm{Ric}(e_3)f + \|A\|^2 f) \leq 0$$

After integration by parts, we have

$$\int_S (\mathrm{Ric}(e_3) + \|A\|^2)f^2 \leq \int_S |\nabla f|^2$$

$$(4.3) \quad \int_S (R - K + \tfrac{1}{2}\|A\|^2)f^2 \leq \int_S |\nabla f|^2$$

For any $t > 0$, let $S_{(t)} = [S \cap (N \backslash N_k)] \cup [S \cap B_t(o)]$. Define a function φ_t by

$$\varphi_t = \begin{cases} 1 & \text{on } S_{(t)} \\ \dfrac{\log \frac{t^2}{r}}{\log t} & \text{on } S_{(t^2)} \setminus S_{(t)} \\ 0 & \text{outside } S_{(t^2)} \end{cases}$$

Then $|\nabla \varphi_t|^2 = \dfrac{|\nabla r|^2}{r^2 (\log t)^2}$ or 0.

Since $|\nabla r|^2$ is bounded, $\int_S |\nabla \varphi_t|^2 \longrightarrow 0$ as $t \longrightarrow \infty$.

Substitute φ_t for f in (4.3) and let $t \longrightarrow \infty$, we have

$$\int_S R - K + \tfrac{1}{2}\|A\|^2 \leq 0 \quad .$$

It implies $\int_S K > 0$ since $R > 0$ on N_k. In fact one should check R, K and $\|A\|^2$ are integrable functions over S. But details are omitted. Q.E.D.

Before giving the proof for $\int_S K \leq 0$, let us consider the topological structure of S from the result $\int_S K > 0$. Since S is noncompact $H_2(S,Z) = 0$ [Vi,p134]. By the Cohn-Vossen inequality $0 < \int_S K \leq 2\pi\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. The first betti number is then equal to 0. From the results of combinatorial topology, we know that the fundamental group $\pi_1(S)$ is free [AS,p102]. Therefore $\pi_1(S) = H_1(S) = 0$ and S is homeomorphic to $\mathbb{R}^2$ [AS,p104].

(ii) <u>Proof of $\int_S K \leq 0$</u>

Let $P_t = \{ x \in \mathbb{R}^3 : (x^1)^2 + (x^2)^2 \leq t^2 \}$. For almost all $t > 0$ ∂P_t intersects S transversally. There is at least one closed curve C in this intersection. Let D_t be a connected component of $S \cap [(N \setminus N_k) \cup P_t]$ with the closed curve C as a component of ∂D_t. We choose D_t so that $D_{t_1} \supset D_{t_2}$ if $t_1 > t_2$. Thus D_t form an exhaustion of S.

By the Gauss-Bonnet theorem, $\int_{D_t} K = 2\pi - \int_{\partial D_t} k$ where k is the geodesic curvature of ∂D_t relative to the inner normal. We choose an orthonormal frame e_1, e_2, e_3 so that e_1 is tangent to ∂D_t, e_2 is the inner normal to ∂D_t and e_3 is the unit normal of S.

Claim : $\int_{\partial D_t} k \geq 2\pi - o(1)$. If the claim is true then the theorem follows immediately.

Proof of Claim. Let $r' = ((x^1)^2 + (x^2)^2)^{\frac{1}{2}}$. On D_t, r' is a constant. Therefore $<e_1, Dr'> = 0$ and

$$0 = e_1<e_1, Dr'> = <D_{e_1} e_1, Dr'> + <e_1, D_{e_1} Dr'>.$$

Since $Dr' = \frac{x'}{t} + O(\frac{1}{t})$ where $x' = \sum_{j=1}^{2} x^j \frac{\partial}{\partial x^j}$ and

$D_{e_1} e_1 = ke_2 - h_{11}$, we have

$$(4.4) \qquad k<e_2, \frac{x'}{t}> + t^{-1} = h_{11}<e_3, \frac{x'}{t}> + \frac{1}{\sigma} <e_1, \frac{\partial}{\partial x^3}>^2 + O(\frac{1}{t}) \| D_{e_1} e_1 \| + O(\frac{1}{t^2})$$

It can be shown that the integral of the right hand side over ∂D_t is equal to $O(\frac{1}{t})$. From (4.4) we have

$$\int_{\partial D_t} k<e_2, \frac{x'}{t}> + \frac{1}{\sigma} L(\partial D_t) = O(t^{-1})$$

Also we can show that $1 + <\frac{x'}{t}, e_2> = o(1)$ on D_t as $t \dashrightarrow \infty$.

Moreover the projection of ∂D_t onto the x^1x^2-plane is a circle of radius t centered at 0, so

$$L(\partial D_t) \geq 2t\pi - o(1).$$

Thus
$$\int_{\partial D_t} k = \int_{\partial D_t} k(1 + <\frac{x'}{t} , e_2>) - k<\frac{x'}{t}, e_2>$$
$$\geq o(1) \int_{\partial D_t} k + 2\pi - o(1)$$

It implies that $\int_{\partial D_t} k \geq 2\pi - o(1)$. Q.E.D.

This complete the proof of Theorem A.

Remark on the proof of $\int_S K \geq 0$. In the original paper by Schoen and Yau, two proofs were given when proving $\int_S K \geq 0$. The second proof in their paper is presented here.

§ 4.2 Proof of Theorem B

Let (N,g) be an asymptotically flat 3-manifold. Suppose for some end, N_k has zero total mass. By (4.1) $g_{ij} = \delta_{ij} + p_{ij}$ on N_k. Therefore $\Gamma^k_{ij} = O(r^{-3})$ and $R_{ijkl} = O(r^{-4})$ on N_k.

Since N is a 3-manifold, it suffices to show that g is Ricci flat. By throwing away the other ends outside a convex ball, we may assume N has only one end or $N \setminus N_k$ is compact with smooth boundary. We require the mean curvature on ∂N is positive.

We shall prove that if $M = 0$, $R \geq 0$ and R is not identically zero, we can always conformally deform the metric g to another metric which is asymptotically flat, scalar flat but has negative total mass. By Theorem A, R is forced to be identically zero. If $R \equiv 0$ and the Ricci curvature is not identically zero, we can find a metric, which is not conformal, but which is again asymptotically flat, scalar flat and has negative total mass. Then by Theorem A again, the Ricci curvature is identically zero.

Lemma 4.1 There is a constant $c_1 > 0$ so that for any function u with compact support on $N \cup \partial N$, we have

$$\left(\int_N u^6 \right)^{\frac{1}{3}} \leq c_1 \int_N |Du|^2 .$$

Note that we do not require u to be zero on ∂N.

In general, this lemma does not hold for complete Riemannian manifolds. The main point in the proof is that g is uniformly equivalent to the Euclidean metric in N_k. Therefore Sobolev lemma can be applied in our case.

We study a differential equation of the form :

$$(4.5) \qquad \triangle v - fv = h \qquad \text{on } N$$

where on N_k $\quad |f| \le \frac{k_5}{1+r^4}$, $|h| \le \frac{k_5}{1+r^4}$,

$|Df| \le \frac{k_6}{1+r^5}$, $|Dh| \le \frac{k_6}{1+r^5}$.

f_+ and f_- will denote the positive and negative parts of f. So $f = f_+ - f_-$ and $|f| = f_+ + f_-$. Let n be the outward unit normal vector to ∂N.

Lemma 4.2 There is a number $\varepsilon_o > 0$ depending on N, k_1, k_2, k_3 and k_4 so that if $(\int_N f_-^{\frac{3}{2}})^{\frac{2}{3}} \le \varepsilon_o$ then (4.5) has a unique solution v on N satisfying $\frac{\partial v}{\partial n} = 0$ on ∂N and v has the expression

$$(4.6) \qquad v = \frac{A}{r} + w \qquad \text{where} \qquad A = -\frac{1}{4\pi}\int_N fv + h \qquad \text{and}$$

$|w| \le \frac{k_7}{1+r^2}$, $|Dw| \le \frac{k_8}{1+r^3}$, $|D^2w| \le \frac{k_9}{1+r^4}$. k_7, k_8, k_9 depend on k_1, k_2, k_3 and N only.

Proof. Existence. Suppose $N_k \approx R^3 \setminus B_\sigma(o)$. For any $t > \sigma$, consider the following boundary value problem :

$$(4.7) \qquad \begin{cases} \triangle v_t - fv_t = h & \text{on } N^t = (N \setminus N_k) \cup (B_t(0) \cap N_k) \\ v_t = 0 & \text{on } \partial B_t(0) \\ \frac{\partial v_t}{\partial n} = 0 & \text{on } \partial N \end{cases}$$

When $\varepsilon_o < \frac{1}{3} c_1^{-}$, where c_1 is the constant in Lemma 4.1, it can be shown that the elliptic operator has trivial kernal. Therefore there exists a solution v_t to the problem (4.7). By local estimates, for any $0 < \alpha < 1$, the $C^{2+\alpha}$ norm of v_t , $t > \sigma$ is bounded. Hence there exists a subsequence converge in C^2 norm on compact subset to a function v. v is then a solution of (4.5).

Proof of the expression (4.6) is omitted.

Uniqueness. Let v, $\bar{v}$ are solutions of (4.5) and $u = v - \bar{v}$. u satisfies

(4.8) $\quad \Delta u - fu = 0, \quad u = O(r^{-1})$ in N , $\quad \frac{\partial u}{\partial n} = 0$ on N.

For all $s > 0$, let $E_s = \{ x \in N : u(x) \geq s \}$. By (4.8) E_s is compact and $\int_{E_s} u \Delta u = \int_{E_s} fu^2$

Using integration by parts, we have

$$\begin{aligned} \int_{E_s} |Du|^2 &= -\int_{E_s} fu^2 \\ &\leq \int_{E_s} u^2 f_- \leq \left(\int_{E_s} f_-^{\frac{3}{2}}\right)^{\frac{2}{3}} \left(\int_{E_s} u^6\right)^{1/3} \\ &\leq \varepsilon_0 \left(\int_{E_s} u^6\right)^{1/3} \end{aligned}$$

By Lemma 4.1

$$\left(\int_{E_s} (u - s)^6\right)^{1/3} \leq c_1 \int_{E_s} |Du|^2 \leq \frac{1}{3} \left(\int_{E_s} u^6\right)^{1/3}$$

By letting $s \longrightarrow 0$, we conclude that $u \leq 0$. Similarily we also prove that $u \geq 0$. This completes the proof of the lemma. Q.E.D.

Lemma 4.3 Suppose g is an asymptotically flat metric on N and R is the scalar curvature of g. ε_0 is the constant defined in Lemma 4.2. If R satisfies

$$\frac{1}{8} \left(\int_N (R_-)^{3/2} \right)^{2/3} \leq \varepsilon_0$$

then there exists a unique function u such that the conformal metric $\bar{g} = u^4 g$ is asymptotically flat, scalar flat and has negative total mass. In fact, the new total mass $\bar{M} = -\frac{1}{16\pi}\int_N Ru$.

Proof. By Lemma 4.2, there exists unique function v satisfies

$$\begin{cases} \Delta v - \frac{1}{8} Rv = \frac{1}{8} R & \text{in } N \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial N \end{cases}$$

and the expression (4.6). Take u = v + 1. u satisfies

$$(4.9) \qquad \begin{cases} \Delta u - \frac{1}{8} Ru = 0 & \text{on } N \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial N \end{cases}$$

Since $v = O(r^{-1})$, u is asymptotic to 1. To show u is positive, let $E = \{ x \in N : u(x) \leq 0 \}$ which is a compact set. By (4.9)

$$\int_E |Du|^2 = -\frac{1}{8} \int_E Ru^2 \leq \cdot \frac{1}{8} \int_E (R_-)u^2 \leq \varepsilon_0 (\int_E u^6)^{1/3}$$

Hence by Lemma 4.1 $(\int_E u^6)^{1/3} \leq c_1 \int_E |Du|^2 \leq \frac{1}{3} (\int_E u^6)^{1/3}$

It deduces a contradiction and forces $u \geq 0$ on N. Since $R = O(r^{-4})$ we can find a constant $K \geq 0$ such that $R + K \geq 0$. Then u satisfies

$$\Delta u - \frac{R + K}{8} u = -\frac{K}{8} u \leq 0$$

By Hopf's maximum principle [GT,p35], u > 0 on N. Since

$$u = 1 + \frac{A}{r} + O(r^{-2}), \text{ if } \bar{g} = u^4 g,$$

$$\bar{g}_{ij} = (1 + \frac{A}{r})^4 \delta_{ij} + O(r^{-2}).$$

So $\bar{M} = -\frac{1}{16\pi} \int_N Ru$. Let H and $\bar{H}$ be the mean curvature relative to g and $\bar{g}$ respectively. By the formula

$$\bar{H} = \frac{1}{u^2} (H + \frac{2}{u} \frac{\partial u}{\partial n})$$

$\bar{H}$ is also positive. Q.E.D.

Corollary 4.4 If M = 0, $R \geq 0$ and R is not identically zero, then there is a metric conformally equivalent to g which is asymptotically flat and N_k has negative total mass.

By Lemma 4.3, Corollary 4.4 follows immediately. By Theorem

A and Corollary 4.4, R is forced to be identically zero.

Define a family of metrics

$$ {}^t g = ({}^t g_{ij}) = (g_{ij} + tR_{ij}) $$

where R_{ij} is the Ricci tensor of g and t is sufficiently small. Let ${}^tR$ be the scalar curvature relative to ${}^tg$. By equation (3.35)

$$ \frac{d}{dt}\,{}^tR\Big|_{t=0} = -\Delta R + \sum_{ij} D_i D_j R^{ij} - \| Ric \|^2 $$

where $\| Ric \|^2 = g^{ik} g^{jl} R_{kl} R_{ij}$. By the second Bianchi identity, one can show that $2 \sum_{ij} D_i D_j R^{ij} = \Delta R$. Since R is identically zero, if we denote $R'_o = \frac{d}{dt}\,{}^tR\big|_{t=0}$ we have

(4.10) $\quad R'_o = - \| Ric \|^2 < 0$

Moreover when t is small enough R satisfies

$$ \frac{1}{8} \left(\int_N ({}^tR_-)^{3/2} \right)^{2/3} \leq \varepsilon_o $$

By Lemma 4.2, there is a positive function u_t so that $u_t\,{}^tg$ is asymptotically flat and scalar flat. Its mass is given by

$$ M_t = - \frac{1}{16\pi} \int_N {}^tRu_t \, dv_t $$

where dv_t is the volume element for ${}^tg$.

<u>We claim that $\frac{d}{dt} M_t\big|_{t=0}$ exists and if $\| Ric\|^2 \neq 0$, $\frac{d}{dt} M_t\big|_{t=0} > 0$.</u> Suppose it is true, for some $t < 0$, $M < 0$. therefore $\bar{g} = u_t^{4}\,{}^tg$ is an asymptotically flat metric which is scalar flat and has negative total mass. By Theorem A, $\| Ric \| = 0$ and g is flat.

<u>Proof of the claim.</u> Let Δ_t be the Laplacian for the metric ${}^tg$. Take $u_o = 1$, $R_o = R = 0$. Define

$$ \Delta^{(h)} = \frac{1}{h}(\Delta_h - \Delta_o) \quad , \qquad u^{(h)} = \frac{1}{h}(u_h - u_o) \quad , $$

$$ R^{(h)} = \frac{1}{h}(R_h - R_o) \quad . $$

u satisfies $\quad \triangle_o u^{(h)} - \frac{1}{8} R_o u^{(h)} = -\triangle^{(h)} u_h + \frac{1}{8} R^{(h)} u_h$

By Lemma 4.2, we have a $C^{2+\alpha}$ bound for $u^{(h)}$ on compact subset of N. The bound depends on C^1 bound of R_o and $-\triangle^{(h)} u_h + \frac{1}{8} R^{(h)} u_h$ which are idependent of h. Therefore there exists a sequence $\{h_i\}$ $h_i \longrightarrow 0$ as $i \longrightarrow$ and for all $0 < \beta < \alpha$ $u^{(h_i)}$ converges in $C^{2+\beta}$ norm on compact subsets to a $C^{2+\alpha}$ function u'_o. u'_o satisfies

$$\triangle_o u'_o - \frac{1}{8} R_o u'_o = -\triangle'_o u_o + \frac{1}{8} R'_o u_o$$

where $\triangle'_o = \frac{d}{dt} \triangle_t \big|_{t=0}$. The uniqueness part of Lemma 4.2 implies u'_o is independent of the sequence $\{h_i\}$ chosen. Thus $\frac{d}{dt} u_t \big|_{t=0}$ exists and equals to u'_o. By the dominated convergence theorem, we conclude that $\frac{d}{dt} M_t \big|_{t=0}$ exists and

$$\begin{aligned} M' &= \frac{d}{dt} M_t \big|_{t=0} \\ &= -\frac{1}{16\pi} \int_N R_o (u_t \, dv_t)' - \frac{1}{16\pi} \int_N R'_o u_o \, dv_o \end{aligned}$$

Since $R_o = 0$ and $u_o = 1$, by (4.10),

$$M' = \frac{1}{16\pi} \int_N \| Ric \|^2$$

This proves our claim and completes the proof of Theorem B.

Q.E.D.

Chapter 5 Related problems

§ 5.1 Prescribing scalar curvature on complete manifolds

The problem of prescribing scalar curvature on compact manifolds was settled by Kazdan and Warner using the following tricks [KW]. Instead of considering pointwise conformal deformation, they consider a metric $\bar{g}$ on M satisfying

(5.1) $$\varphi^*(\bar{g}) = u^{\frac{4}{n-2}} g$$

where φ is a diffeomorphism of M onto itself and u is a positive function. If R and $\bar{R}$ are scalar curvatures of g and $\bar{g}$ respectively, then

(5.2) $$\frac{4(n-1)}{n-2}\Delta u - Ru = -(\bar{R}\circ\varphi)u$$

They studied the first eigenvalue $\lambda_1(g)$ of the elliptic operator

$$L(u) = \Delta u - \frac{n-2}{4(n-1)} Ru \quad .$$ By Proposition 1.3 , the sign of $\lambda_1(g)$ is a conformal invariant. Kazdan and Warner defines a function $\bar{R}$ to be in CE(g) if there exist a diffeomorphism φ and a metric g satisfying (5.1), (5.2).

Theorem 5.1 Let M be a compact manifold of dimension ≥ 3.

(i) If $\lambda_1(g) < 0$, any function $\bar{R}$ which is negative somewhere is in CE(g).

(ii) If $\lambda_1(g) = 0$, any function $\bar{R}$ which changes sign or is identically zero is in CE(g).

(iii) If $\lambda_1(g) > 0$, and there is a constant function R_1 in CE(g), then any function $\bar{R}$ which is positive somewhere is in CE(g).

Since the Yamabe problem is solved, the condition in part (iii) of Theorem 5.1 always holds. By Aubin's result [Au1,p130], we can always find a metric g_1 such that $\lambda_1(g_1) < 0$. Then we obtain :

<u>Theorem 5.2</u> Let M be a compact manifold of dimension ≥ 3.

(i) Every function which is negative somewhere is the scalar curvature of some metric.

(ii) If M admits a metric g such that $\lambda_1(g) > 0$, then every function is the scalar curvature of some metric.

Except for some results on $\mathbb{R}^n$ and manifolds with nonnegative scalar curvature, little is known about pointwise conformal deformation of metrics on complete open manifolds. In general, imbedding theorems do not apply to non-compact manifolds. The Kondrakov theorem does not hold on non-compact manifolds while the Sobolev theorem holds in case of positive injectivity radius. Therefore we cannot just follow the proof in the compact case.

In many cases, the differential equation (1.1) is not uniformly elliptic on a non-compact manifold. For example, consider the Poincare disk (D,g):

$$D = \left\{ x \in \mathbb{R}^n,\ |x| < 1 \right\} \quad \text{and} \quad g_{ij} = 4(1 - |x|^2)^{-2}\, \delta_{ij} \,.$$

(D,g) has constant sectional curvature -1 and scalar curvature -n(n-1). Equation (1.1) becomes :

$$(5.3) \qquad \frac{(1-|x|^2)^2}{4}\Delta u + n(n-1)u = -\,Ru$$

where Δ is the ordinary Euclidean Laplacian. (5.3) is clearly not uniformly elliptic.

One usually requires the conformal metric to be complete. It is not an easy task to check whether the solution of (1.1) provides a complete metric or not.

(a) Euclidean spaces.

On $\mathbb{R}^n$ equation (1.1) becomes

$$\Delta u = - Ru$$

where we write R for $\bar{R}$. In his paper [Ni2], Ni studied this differential equation and showed that if a function R on $\mathbb{R}^n$ decays in an appropriate rate, one can find infinitely many complete metrics which are conformal to the usual metric and have R as their scalar curvature.

Theorem 5.3 [Ni2,p527] (We write $x = (x_1, x_2)\ \mathbb{R}^{n-m} \times \mathbb{R}^m = \mathbb{R}^n$)
If $m \geq 3$ and $|R(x_1, x_2)| \leq c|x_2|^{-\ell}$ for $|x_2|$ large and uniformly in $x_1 \in \mathbb{R}^{n-m}$ for some constant $c > 0$ and $\ell > 2$, there exists infinitely many Riemannian metrics g on $\mathbb{R}^n$ with the following properties :

(i) R is the scalar curvature of g;

(ii) g is conformal to the usual metric;

(iii) g is complete.

Ni pointed out that this theorem can be extended to more general manifolds other than $\mathbb{R}^n$: when M is a Riemannian manifold with zero scalar curvature, Theorem 5.3 also holds for $M \times R^m$ when $m \geq 3$.

<u>Theorem 5.4 [Ni2, p528]</u> Let $R \geq 0$ be a radial symmetric function on R. We also write $R(s)$ for $R(x)$ when $|x| = s$. If

$$\int_0^r s^n R'(s)\, ds \leq 0 \qquad \text{for all } r \geq 0,$$

there exists infinitely many Riemannian metric g such that each one of them is conformal to the usual metric on $\mathbb{R}^n$ and has R as its scalar curvature. But g may not be complete.

Ni also gave some nonexistence theorems. For any positive function R on $\mathbb{R}^n$, let

$$\hat{R}(r) = \left(\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{ds}{R(x)^{\frac{n-2}{4}}} \right)^{-\frac{4}{n-2}}$$

<u>Theorem 5.5 [Ni2, p528]</u> Let $R \geq 0$ in $\mathbb{R}^n$ outside a compact set. If $\hat{R} \geqslant c|x|^2$ at infinity, there is no metric conformal to the usual metric on R^n and has R as its scalar curvature.

<u>Theorem 5.5′ [Ni2, p528]</u> If $R \leq 0$ in $\mathbb{R}^n$ and $\widehat{(-R)} \geq C|x|^{-k}$ at infinity for some constant $k < 2$, then there is no metric conformal to the usual metric on $\mathbb{R}^n$ and has R as its scalar curvature.

(b) <u>Complete manifolds with negative curvature.</u>

Aviles and McOwen [AM] studied the question of prescribing scalar curvature on complete manifolds of negative sectional curvature. Using the method of upper and lower solutions, they obtained the following general theorem and got a result on the Poincare disk as a corollary.

Theorem 5.6 [AM,p276] Let (M,g) be a simply connected complete Riemannian manifold of dimension ≥ 3 with sectional curvature K satisfying $-A \leq K \leq -B$ for some positive constants A,B satisfying
$$1 \leq \left(\frac{A}{B}\right)^2 < \frac{(n-1)^2}{n(n-2)} .$$
Let $\overline{R}$ be a function on M such that $-a \leq \overline{R}(x) \leq -b < 0$ for $x \in M \setminus M_o$ where M_o is a compact set and $a \geq b > 0$ are constants. Then there exists $\xi > 0$ such that (1.1) has a solution which is bounded between two positive constant provided $\overline{R}(x) \leq \xi$ for all $x \in M$. If $\overline{R}(x) \leq 0$ for all $x \in M$, then the solution is unique.

Corollary 5.7 [AM,p280] Let R be a function on the Poincare disk (D,g). If $R(x) = -n(n-1) + H(x)$ where
$$\sup \left\{(1 - |x|)^t H(x) : x \in D\right\} < \infty$$
for some $t > 0$, then there exists unique solution u of (5.3) such that $\lim_{|x| \to 1} u(x) = 1$.

5.2 The Yamabe problem on CR manifolds

Let M be a manifold of dimension $2n+1$. Suppose there exists a subbundle $H(M)$ of the tangent bundle of M, each fiber of which has a structure of a complex n-dimensional vector space. Then M is said to have a CR structure. We can write

$$H(M) \otimes \mathbb{C} \;=\; T_{1,0} \oplus \overline{T}_{1,0}$$

where $T_{1,0} \cap \overline{T}_{1,0} = 0$. We assume that the CR structure is integrable, that is, the Lie bracket $[\, T_{1,0}\, ,\, T_{1,0}\,] \subset T_{1,0}$. If θ is a global nonvanishing 1 form on M annihilating H(M) then the pair (M, θ) is called a pseudo-hermitian manifold. The Levi form is a hermitian form on $T_{1,0}$ given by

$$\langle Z, W\rangle_\theta \;=\; -id\theta(Z, w) \;, \quad Z, W \in T_{1,0} \;.$$

M is strictly pseudoconvex if the levi form is positive definite.

Webster [We,p29] has shown that one can define a intrinsic connection on M as follows. On a pseudo-hermitian manifold M, we can choose $\theta^\alpha \in T_{1,0}$ so that $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ form a local basis of $\mathbb{C}T^*M$ and

$$d\theta \;=\; i\, g_{\alpha\bar\beta}\; \theta^\alpha \wedge \theta^{\bar\beta}$$

Then there exists uniquely defined ω_β^α , τ^α satisfying

(5.4) $\quad d\theta^\alpha = \theta^\beta \wedge \omega_\beta^\alpha + \theta \wedge \tau^\alpha$

such that $\tau^\alpha = 0 \mod \theta, \theta^\beta\ \theta^{\bar\beta}$ and

(5.5) $\quad dg_{\alpha\bar\beta} - \omega_\alpha^\gamma\, g_{\gamma\bar\beta} - g_{\alpha\bar\gamma}\, \omega_{\bar\beta}^{\bar\gamma} = 0$

By differentiating (5.4) and (5.5) we have

(5.6) $\quad 0 = \theta^\beta \wedge [\, d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha - i\,\theta_\beta \wedge \tau^\alpha\,] + \theta \wedge [\, d\tau^\alpha - \tau^\beta \wedge \omega_\beta^\alpha\,]$

(5.7) $\quad 0 = (d\omega_\alpha^\gamma - \omega_\alpha^\mu \wedge \omega_\mu^\gamma)\, g_{\gamma\bar\beta} + g_{\alpha\bar\gamma}\,(d\omega_{\bar\beta}^{\bar\gamma} - \omega_{\bar\beta}^{\bar\mu} \wedge \omega_{\bar\mu}^{\bar\gamma})$

We define

$$\Omega_\beta^\alpha = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha - i\,\theta_\beta \wedge \tau^\alpha + i\,\tau_\beta \wedge \theta^\alpha$$

$$\Omega^\alpha = d\tau^\alpha - \tau^\beta \wedge \omega_\beta^\alpha$$

By (5.6) and (5.7) we get

(5.8) $\quad \theta^\beta \wedge \Omega_\beta^\alpha + \theta \wedge \Omega^\alpha = 0$

(5.9) $\quad \Omega_{\bar\alpha\beta} + \Omega_{\beta\bar\alpha} = 0$

(5.9) implies that $\Omega_{\beta\bar{\alpha}} = \chi_{\beta\bar{\alpha}\rho} \wedge \theta^{\rho} + \lambda_{\beta\bar{\alpha}} \wedge \theta$

for certain 1 forms $\chi_{\beta\bar{\alpha}\rho}$ and $\lambda_{\beta\bar{\alpha}}$. We may assume they contain no terms in θ . Then it can be shown that

$$\chi_{\beta\bar{\alpha}\rho} = -R_{\beta\bar{\alpha}\rho\bar{\sigma}}\,\theta^{\bar{\sigma}}$$

and $\qquad \lambda_{\beta\bar{\alpha}} = W_{\beta\bar{\alpha}\gamma}\,\theta^{\gamma} - W_{\bar{\alpha}\beta\bar{\gamma}}\,\theta^{\bar{\gamma}}$

where $\quad R_{\beta\bar{\alpha}\rho\bar{\sigma}}$ and $W_{\beta\bar{\alpha}\gamma}$ satisfy

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho}$$

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}$$

$$W_{\beta\bar{\alpha}\gamma} = W_{\gamma\bar{\alpha}\beta}\,.$$

Therefore $\quad \Omega_{\beta}{}^{\alpha} = R_{\beta}{}^{\alpha}{}_{\rho\bar{\sigma}}\,\theta^{\rho} \wedge \theta^{\bar{\sigma}} + W_{\beta}{}^{\alpha}{}_{\rho}\,\theta^{\rho} \wedge \theta - W^{\alpha}{}_{\beta\bar{\sigma}}\,\theta^{\bar{\sigma}} \wedge \theta$

By (5.8) we also obtain

$$\Omega^{\alpha} = W^{\alpha}{}_{\rho\bar{\sigma}}\,\theta^{\rho} \wedge \theta^{\bar{\sigma}} - A^{\alpha}{}_{\bar{\gamma}}\,\tau^{\bar{\gamma}} \wedge \theta + B^{\alpha}{}_{\bar{\sigma}}\,\theta^{\bar{\sigma}} \wedge \theta$$

where $B^{\alpha}{}_{\bar{\sigma}}$ satisfies $B_{\beta\gamma} = B_{\gamma\beta}$. The tensor $R_{\beta\bar{\alpha}\rho\bar{\sigma}}$ is called the curvature tensor. Then we define the Ricci tensor and the scalar curvature :

$$R_{\rho\bar{\sigma}} = R_{\alpha}{}^{\alpha}{}_{\rho\bar{\sigma}} \quad , \quad R = g^{\rho\bar{\sigma}} R_{\rho\bar{\sigma}}\,.$$

By strictly pseudoconvexity of M, there is a natural volume form of M, namely $\theta \wedge (d\theta)^n$. The sublaplacian operator Δ_b is defined on functions by

$$\int_M (\Delta_b u)\, v \;\theta \wedge (d\theta)^n = \int_M \langle du, dv \rangle_{\theta}\; \theta \wedge (d\theta)^n.$$

for all $v \in C_0^{\infty}(M)$. Under a change of pseudo-hermitian structure

$$\bar{\theta} = u^{\frac{2}{n}}\,\theta$$

where u is a positive function, Lee [Le,p24] calculated that the scalar curvature transforms by the formula :

(5.5) $\qquad u^{1-\frac{2}{n}}\,\bar{R} = \dfrac{2(n+1)}{n}\,\Delta_b u + Ru$

where $\bar{R}$ is the scalar curvature relative to $\bar{\theta}$ .Hence the problem of deforming the pseudo-hermitian structure to a structure of constant scalar curvature is equivalent to solving the differential equation (5.5), when $\bar{R}$ is a constant.

Jerison and Lee [JL,p58] got some results which are analogous to Proposition 1.1, 1.3 and Theorem 2.2.

Theorem 5.8 Let M be a compact strictly pseudoconvex, integrable CR manifold of dimension 2n+1. Let

$$E(u) = \int_M \left(\frac{2(n+1)}{n} |Du|_\theta^2 + Ru \right) \theta \wedge (d\theta)^n \quad , \text{ and}$$

$$(5.6) \qquad \lambda_M = \inf \left\{ E(u) : u \geq 0 \text{ and } \int_M u^{\frac{2(n+1)}{n}} \theta \wedge (d\theta)^n = 1 \right. .$$

Then (i) λ_M depends on the CR structure but not on the pseudo-hermitian structure.

(ii) $\lambda_M \leq \lambda_{S^{2n+1}}$.

(iii) If $\lambda_M < \lambda_{S^{2n+1}}$, then the infimum in (5.6) is attained by a positive smooth function.

However, the solution of the corresponding Yamabe problem is still open. It is not clear what should be the analogue of the positive mass theorem.

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